Analytic black hole perturbation approach to gravitational radiation

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Abstract

We review analytic methods to perform the post-Newtonian expansion of gravitational waves induced by a particle orbiting a massive compact body, based on the black hole perturbation theory. There exist two different methods of the post-Newtonian expansion. Both are based the Teukolsky equation. In one method, the Teukolsky equation is transformed into a Regge-Wheeler type equation that reduces to the standard Klein-Gordon equation in the flat space limit, while in the other method, which were introduced by Mano, Suzuki and Takasugi relatively recently, the Teukolsky equation is directly used in its original form. The former has an advantage that it is intuitively easy to understand how various curved space effects come into play. However, it becomes increasingly complicated when one goes on to higher and higher post-Newtonian orders. In contrast, the latter has an advantage that a systematic calculation to higher post-Newtonian orders is relatively easily implementable, but otherwise so mathematical that it is hard to understand the interplay of higher order terms. In this paper, we review both methods so that their pros and cons may be clearly seen. We also review some results of calculations of gravitational radiation emitted by a particle orbiting a black hole.

1 Introduction

1.1 General

In the past several years, there has been substantial progress in the projects of ground-based laser interferometric gravitational wave detectors which include LIGO [1], VIRGO [2], GEO600 [3], and TAMA300 [75, 9, 4]. TAMA300 has been in operation since 1999. LIGO and GEO600 started to operate in 2002. VIRGO is expected to start operation in 2003. As another type of detectors, resonant bar detectors [8] have been operating continuously for the past several

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years. There are several future projects as well. Most importantly, the Laser Interferometer Space Antenna (LISA) project is in progress [5, 6]. For a review of ground and space laser interferometers, see, e.g., [62].

The detection of gravitational waves will be done by extracting gravitational wave signals from noisy data stream. In developing the data analysis strategy, the detailed knowledge of the gravitational wave forms will help us greatly to detect a signal, and to extract the physical information of its source. Thus, it has become a very important problem for theorists to predict with sufficiently good accuracy the wave forms from possible gravitational wave sources.

Gravitational waves are generated by dynamical astrophysical events and they are expected to be strong enough to be detected when compact stars such as neutron stars or black holes are involved in such events. In particular, coalescing compact binaries are considered to be the most promising sources of gravitational radiation that can be detected by the ground-based laser interferometers. The last inspiral phase of a coalescing compact binary, in which the binary stars orbit each other for $\sim 10^4$ cycles, will be in the bandwidth of the interferometers, and it may be not only detected but also provides us important astrophysical information of the system by comparing the inspiral waveform with theoretical templates if they have sufficient accuracy. Such an event may be detected by the initial phase interferometers if we are lucky, and it will be detected by the various advanced detectors [49, 55].

To predict the wave forms, a conventional approach is to formulate the Einstein equations with respect to the flat Minkowski background and apply the post-Newtonian expansion to the resulting equations (see the subsection below).

In this paper, however, we review a different approach, namely the black hole perturbation approach. In this approach, binaries are assumed to consist of a massive black hole and a small compact star which is taken to be a point particle. Hence, its applicability is constrained to the case of binaries with large mass ratio. Nevertheless, there are several advantages that cannot be overlooked.

Most importantly, the black hole perturbation equations take full account of general relativistic effects of the background spacetime and they are applicable to arbitrary orbits of a small mass star. In particular, if a numerical approach is taken, gravitational waves from highly relativistic orbits can be calculated. Then, if we can develop a method to calculate gravitational waves to a sufficiently high PN order analytically, it can not only give insight into how and when general relativistic effects become important, by comparing with numerical results, but also give us a knowledge complementary to the conventional post-Newtonian approach, about such as yet-unknown higher PN order terms or general relativistic spin effects.

Moreover, one of the main targets of LISA is to observe phenomena associated with formation and evolution of super-massive black holes in galactic centers. In particular, a gravitational wave event of a compact star spiraling into such a super-massive black hole is indeed the case for the black hole perturbation theory.

1.2 Post-Newtonian expansion of gravitational waves

The post-Newtonian expansion of general relativity assumes that the internal gravity of a source is small so that the deviation from the Minkowski metric is small, and that velocities associated with the source are small compared to the speed of light c. When we consider the orbital motion of a compact binary system, these two conditions become essentially equivalent to each other. Although, both conditions may be violated inside each of the compact objects, this is not regarded as a serious problem of the post-Newtonian expansion as long as we are concerned with gravitational waves generated from the orbital motion, and the two bodies are usually assumed to be point-like objects in the calculation. In fact, recently Itoh, Futamase, and Asada [37, 38] developed a new post-Newtonian method which can deal with a binary system in which the constituent bodies may have strong internal gravity, based on earlier work by Futamase and Schutz [31, 32]. They derived the equations of motion to 2.5PN order and obtained a complete agreement with the Damour-Deruelle equations of motion [24, 25], which assumes the validity of the point-particle approximation.

There are two existing approaches of the post-Newtonian expansion to calculate gravitational waves: one developed by Blanchet, Damour, and Iyer [15, 12] and another by Will and Wiseman [84] based on previous work by Epstein, Wagoner, and Will [27, 81]. In both approaches, the gravitational waveforms and luminosity are expanded in time derivatives of radiative multipoles, which are then related to some source multipoles (the relation between them contains the "tails") which are expressed as integrals over the matter source and the gravitational field. The source multipoles are combined with the equations of motion to obtain explicit expressions in terms of the source masses, positions, and velocities.

One issue of the post-Newtonian calculation arises from the fact that the post-Newtonian expansion can be applied only to the near zone field of the source. In the conventional post-Newtonian formalism, the harmonic coordinates are used to write down the Einstein equations. If we define the deviation from the Minkowski metric as

$$h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{1}$$

the Einstein equations are schematically written in the form

$$\Box h^{\mu\nu} = 16\pi|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h), \tag{2}$$

together with the harmonic-gauge condition, $\partial_\nu h^{\mu\nu} = 0$, where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, is the D'Alambertian operator in flat space time, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and $\Lambda^{\mu\nu}(h)$ represents the non-linear terms in the Einstein equations. The Einstein equations (2) are integrated using the flat space retarded integrals. In order to perform the post-Newtonian expansion, if we naively expand the retarded integrals in power of $1/c$, there appears divergent integrals. This is a technical problem which arises due to the near zone nature of the post-Newtonian approximation. In the BDI approach, in order to integrate the retarded integrals, and

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to evaluate the radiative multipole moments at infinity, two kind of approximation method are introduced. One is the multipolar post-Minkowski expansion, which can be applied to a region outside the source including infinity, and the other is the near zone, post-Newtonian expansion. These two expansions are matched in the intermediate region where both expansions are valid, and the radiative multipole moments are evaluated at infinity. In the WW approach, the retarded integrals are evaluated directly, without expanding in terms of $1/c$, in the region outside the source in a novel way.

The lowest order of the gravitational waves are given by the Newtonian quadrupole formula. It is standard to call the post-Newtonian formulas for the wave forms and luminosity which contain terms up to $O((v/c)^n)$ beyond the Newtonian quadrupole formula as the $n/2$PN formulas. Evaluation of gravitational waves emitted to infinity from a compact binary system has been successfully carried out to the 3.5 post-Newtonian (PN) order beyond the lowest Newtonian quadrupole formula (apart from an undetermined coefficient that appears at 3PN order) in the BDI approach [15, 12]. The computation of the 3.5PN flux requires the 3.5PN equations of motion. See a review by Blanchet [13] for details on post-Newtonian approaches. Up to now, both approaches give the same answer for the gravitational wave forms and luminosity to 2PN order.

1.3   Linear perturbation theory of black hole

In the black hole perturbation approach, we deal with gravitational waves from a particle of mass μ orbiting a black hole of mass M, assuming $\mu \ll M$. The perturbation of a black hole spacetime is evaluated to linear order in μ/M. The equations are essentially in the form of Eq. (2) with $\eta_{\mu\nu}$ replaced by the background black hole metric $g_{\mu\nu}^{BH}$ and the higher order terms $\Lambda(h)_{\mu\nu}$ neglected. Thus, apart from the assumption $\mu \ll M$, the black hole perturbation approach is not restricted to slow-motion sources, nor to small derivations from the Minkowski spacetime, and the Green function used to integrate the Einstein equations contains the whole curved spacetime effect of the background geometry.

The black hole perturbation theory was originally developed as a metric perturbation theory. For non-rotating (Schwarzschild) black holes, a single master equation for the metric perturbation was derived by Regge-Wheeler [61] for the so-called odd parity part, and later by Zerilli [86] for the even parity part. These equations have a nice property that they reduce to the standard Klein-Gordon wave equation in the flat space limit. However, no such equation has been found in the case of a Kerr black hole so far.

Then, based on the Newmann-Penrose null-tetrad formalism, in which the tetrad components of the curvature tensor are the fundamental variables, a master equation for the curvature perturbation was first developed by Bardeen and Press [11] for a Schwarzschild black hole without source ($T^{\mu\nu} = 0$), and by Teukolsky [78] for a Kerr black hole with source ($T^{\mu\nu} \neq 0$). The master equation is called the Teukolsky equation, and it is a wave equation for a null-tetrad component of the Weyl tensor ψ_0 or ψ_4. In the source-free case, Chrzanowski [21]

and Wald [82] developed a method to construct the metric perturbation from the curvature perturbation.

The Teukolsky equation has, however, a rather complicated structure as a wave equation. Even in the flat space limit, it does not reduce to the standard Klein-Gordon form. Then, Chandrasekhar showed that the Teukolsky equation can be transformed to the form of the Regge-Wheeler or Zerilli equation for the source-free Schwarzschild case [19]. A generalization of this to the Kerr case with source was done by Sasaki and Nakamura [66, 67]. They gave a transformation that brings the Teukolsky equation to a Regge-Wheeler type equation that reduces to the Regge-Wheeler equation in the Schwarzschild limit. It may be noted that the Sasaki-Nakamura equation contains an imaginary part, suggesting that either it is unrelated to a (yet-to-be-found) master equation for the metric perturbation for the Kerr geometry or it implies the non-existence of such a master equation.

As mentioned above, an important difference between the black hole perturbation approach and the conventional post-Newtonian approach appears in the structure of the Green function used to integrate the wave equations. In the black hole perturbation, the Green function takes account of the curved space-time effect on the wave propagation, which implies complexity of its structure in contrast to the flat space Green function. Thus, since the system is linear in the black hole perturbation approach, the most non-trivial task is the construction of the Green function.

There are many papers that deal with a numerical evaluation of the Green function and calculations of gravitational waves induced by a particle. See Breuer [18], Chandrasekhar [20], and Nakamura, Oohara and Kojima [48] for reviews and for references on earlier papers.

Here, we are interested in an analytical evaluation of the Green function. One way is to adopt the post-Minkowski expansion assuming $GM/c^2 \ll r$. Note that, for bound orbits, the condition $GM/c^2 \ll r$ is equivalent to the the condition for the post-Newtonian expansion $v^2/c^2 \ll 1$. If we can calculate the Green function to a sufficiently high order in this expansion, we may be able to obtain a rather accurate approximation of it that can be applicable to a relativistic orbit fairly close to the horizon, possibly to the radius as small as the inner-most stable circular orbit (ISCO), which is given by $r_{ISCO} = 6GM/c^2$ in the case of a Schwarzschild black hole.

It turns out that this is indeed possible. Though there arise some complications as one goes to higher PN orders, they are relatively easy to handle as compared to situations one encounters in the conventional post-Newtonian approaches. Thus, very interesting relativistic effects such as tails of gravitational waves can be investigated easily. Further, we can also easily investigate convergence properties of the post-Newtonian expansion by comparing a numerically calculated exact result with the corresponding analytic but approximate result. In this sense, the analytic black hole perturbation approach can provide an important test of the post-Newtonian expansion.

1.4 Brief historical notes

Let us briefly review some of past works on post-Newtonian calculations in the black hole perturbation theory. Although the literature on numerical calculations of gravitational waves emitted by a particle orbiting a black hole is abundant, there are not so many papers that deal with the post-Newtonian expansion of gravitational waves, mainly because it had not been necessary until recently when the construction of accurate theoretical templates for the interferometric gravitational wave detectors had become an urgent issue.

In the case of orbits in the Schwarzschild background, one of the earliest works was done by Gal'tsov, Matiukhin and Petukhov [34], who considered a case when a particle is in a slightly eccentric orbit around a Schwarzschild black hole, and calculated the gravitational waves up to 1PN order. Poisson [56] considered a circular orbit around a Schwarzschild black hole and calculated the wave forms and luminosity to 1.5PN order at which the tail effect appears. Cutler, Finn, Poisson and Sussman [23] also worked on the same problem numerically by using the least square fitting technique to the numerically evaluated data for the luminosity, and obtained a post-Newtonian formula for the luminosity to 2.5PN order. Subsequently, a highly accurate numerical calculation was carried out by Tagoshi and Nakamura [72]. They obtained the formulas for the luminosity to 4PN order numerically by using the least square fitting method. They found the $\log v$ terms in the luminosity formula at 3PN and 4PN orders. They concluded that, although the convergence of the post-Newtonian expansion is slow, the luminosity formula accurate to 3.5PN order will be good enough as theoretical templates for the ground-based interferometers, to represent the orbital phase evolution of coalescing compact binaries. After that, Sasaki [65] found an analytic method and obtained formulas which are needed to calculate the gravitational waves to 4PN order. Then, Tagoshi and Sasaki [73] obtained the gravitational wave forms and luminosity to 4PN order analytically, and confirmed the results of Tagoshi and Nakamura. These calculations were extended to 5.5PN order by Tanaka, Tagoshi, and Sasaki [77].

In the case of orbits around a Kerr black hole, Poisson calculated the 1.5PN order corrections to the wave forms and luminosity due to the rotation of the black hole and showed that the result agrees with the standard post-Newtonian effect due to spin-orbit coupling [57]. Then, Shibata, Sasaki, Tagoshi and Tanaka [69] calculated the luminosity to 2.5PN order. They calculated the luminosity from a particle in circular orbit with small inclination from the equatorial plane. They used the Sasaki-Nakamura equation as well as the Teukolsky equation. This analysis was extended to 4PN order by Tagoshi, Shibata, Tanaka and Sasaki [74] in which the orbit of the test particle was restricted to circular ones on the equatorial plane. The analysis in the case of slightly eccentric orbit on the equatorial plane was also done by Tagoshi [70] to 2.5PN order.

Tanaka, Mino, Sasaki and Shibata [76] considered the case when a spinning particle is in a circular orbit near the equatorial plane of a Kerr black hole, based on the Papaetrou equations of motion for a spinning particle [52] and the energy momentum tensor of a spinning particle by Dixon [26]. They derived

the luminosity formula to 2.5PN order which includes the linear order effect of the particle's spin.

The absorption of gravitational waves into the black hole horizon, appearing at 4PN order in the Schwarzschild case, was calculated by Poisson and Sasaki for a particle in a circular orbit [59]. The black hole absorption in the case of rotating black hole appears at 2.5PN order [33]. Using a new analytic method to solve the homogeneous Teukolsky equation found by Mano, Suzuki and Takasugi [43], the black hole absorption in the Kerr case was calculated by Tagoshi, Mano, and Takasugi [71] to 6.5PN order beyond the quadrupole formula.

If gravity is not described by the Einstein theory but by the Brans-Dicke theory, there will appear scalar type gravitational waves as well as transverse-traceless gravitational waves. Such scalar type gravitational waves were calculated to 2.5PN order by Ohashi, Tagoshi and Sasaki [51] in the case when a compact star is in a circular orbit on the equatorial plane around a Kerr black hole.

In the rest of the paper, we use the units $c = G = 1$.

2 Basic formulas for the black hole perturbation

2.1 Teukolsky formalism

In terms of the conventional Boyer-Lindquist coordinates, the metric of a Kerr black hole is expressed as

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)d\varphi - a\, dt\right]^2 \\
&\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,,
\end{aligned}
\tag{3}
$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. In the Teukolsky formalism [78], the gravitational perturbations of a Kerr black hole are described by a Newman-Penrose quantity $\psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha\bar{m}^\beta n^\gamma\bar{m}^\delta$, where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor and

$$
\begin{aligned}
n^\alpha &= ((r^2 + a^2), -\Delta, 0, a)/(2\Sigma), \\
m^\alpha &= (ia\sin\theta, 0, 1, i/\sin\theta)/(\sqrt{2}(r + ia\cos\theta)).
\end{aligned}
$$

The perturbation equation for $\phi \equiv \rho^{-4}\psi_4$, $\rho = (r - ia\cos\theta)^{-1}$, is given by

$$
{}_s\mathcal{O}\,\phi = 4\pi\Sigma\hat{T}\,.
\tag{4}
$$

Here, the operator ${}_s\mathcal{O}$ is given by

$$
\begin{aligned}
{}_s\mathcal{O} &= -\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2\sin^2\theta\right]\partial_t^2 - \frac{4Mar}{\Delta}\partial_t\partial_\phi - \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\partial_\phi^2 \\
&\quad + \Delta^{-s}\partial_r(\Delta^{s+1}\partial_r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + 2s\left[\frac{a(r - M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\partial_\phi \\
&\quad + 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\partial_t - s(s\cot^2\theta - 1),
\end{aligned}
\tag{5}
$$

with $s = -2$. The source term $\hat{T}$ is given by

$$
\begin{aligned}
\hat{T} &= 2(B_2' + B_2^{*'}), \\
B_2' &= -\frac{1}{2}\rho^8\overline{\rho}L_{-1}[\rho^{-4}L_0(\rho^{-2}\overline{\rho}^{-1}T_{nn})] \\
&\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 L_{-1}[\rho^{-4}\overline{\rho}^2 J_+(\rho^{-2}\overline{\rho}^{-2}\Delta^{-1}T_{\overline{m}n})], \\
B_2'^* &= -\frac{1}{4}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}J_+(\rho^{-2}\overline{\rho}T_{\overline{m}\overline{m}})] \\
&\quad - \frac{1}{2\sqrt{2}}\rho^8\overline{\rho}\Delta^2 J_+[\rho^{-4}\overline{\rho}^2\Delta^{-1}L_{-1}(\rho^{-2}\overline{\rho}^{-2}T_{\overline{m}n})], \quad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
L_s &= \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta, \quad (7) \\
J_+ &= \partial_r + iK/\Delta; \quad K = (r^2 + a^2)\omega - ma,
\end{aligned}
$$

and T_{nn}, $T_{\overline{m}n}$ and $T_{\overline{m}\overline{m}}$ are the tetrad components of the energy momentum tensor ($T_{nn} = T_{\mu\nu}n^\mu n^\nu$ etc.). The bar denotes the complex conjugation.

If we set $s = 2$ in Eq. (4), with appropriate change of the source term, it becomes the perturbation equation for ψ_0. Moreover, it describes the perturbation for scalar field ($s = 0$), neutrino field ($|s| = 1/2$), and electromagnetic field ($|s| = 1$) as well.

We decompose ψ_4 into the Fourier-harmonic components according to

$$
\rho^{-4}\psi_4 = \sum_{\ell m}\int d\omega\, e^{-i\omega t + im\varphi}\,\,_{-2}S_{\ell m}(\theta)R_{\ell m\omega}(r), \quad (8)
$$

The radial function $R_{\ell m\omega}$ and the angular function $_sS_{\ell m}(\theta)$ satisfy the Teukolsky equations with $s = -2$ as

$$
\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right) - V(r)R_{\ell m\omega} = T_{\ell m\omega}, \quad (9)
$$

$$
\begin{aligned}
\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}\right\} - a^2\omega^2\sin^2\theta - \frac{(m - 2\cos\theta)^2}{\sin^2\theta}\right. \\
\left. + 4a\omega\cos\theta - 2 + 2ma\omega + \lambda\right]_{-2}S_{\ell m} = 0. \quad (10)
\end{aligned}
$$

The potential $V(r)$ is given by

$$
V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda, \quad (11)
$$

where λ is the eigenvalue of $_{-2}S_{\ell m}^{a\omega}$. The angular function $_sS_{\ell m}(\theta)$ is called the spin-weighted spheroidal harmonic, which is usually normalized as

$$
\int_0^\pi |_{-2}S_{\ell m}|^2\sin\theta d\theta = 1. \quad (12)
$$

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In the Schwarzschild limit, it reduces to the spin-weighted spherical harmonic with $\lambda \to \ell(\ell+1)$. In the Kerr case, however, no analytic formula for λ is known. The source term $T_{\ell m \omega}$ is given by

$$T_{\ell m \omega} = 4 \int d\Omega dt \rho^{-5} \overline{\rho}^{-1} (B_2' + B_2'^*) e^{-im\varphi + i\omega t} \frac{{}_{-2}S_{\ell m}^{a\omega}}{\sqrt{2\pi}}, \tag{13}$$

We mention that for orbits of our interest, which are bounded, $T_{\ell m \omega}$ has support only in a compact range of r.

We solve the radial Teukolsky equation by using the Green function method. For this purpose, we define two kinds of homogeneous solutions of the radial Teukolsky equation:

$$R_{\ell m \omega}^{\rm in} \quad \to \quad \begin{cases} B_{\ell m \omega}^{\rm trans} \Delta^2 e^{-ikr^*} & \text{for } r \to r_+ \\ r^3 B_{\ell m \omega}^{\rm ref} e^{i\omega r^*} + r^{-1} B_{\ell m \omega}^{\rm inc} e^{-i\omega r^*} & \text{for } r \to +\infty, \end{cases} \tag{14}$$

$$R_{\ell m \omega}^{\rm up} \quad \to \quad \begin{cases} C_{\ell m \omega}^{\rm up} e^{ikr^*} + \Delta^2 C_{\ell m \omega}^{\rm ref} e^{-ikr^*} & \text{for } r \to r_+, \\ C_{\ell m \omega}^{\rm trans} r^3 e^{i\omega r^*} & \text{for } r \to +\infty \end{cases} \tag{15}$$

where $k = \omega - ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\begin{aligned} r^* &= \int \frac{dr^*}{dr} dr \\ &= r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \end{aligned} \tag{16}$$

where $r_\pm = M \pm \sqrt{M^2 - a^2}$, and we have fixed the integration constant.

Combining with the Fourier mode $e^{-i\omega t}$, we see that $R_{\ell m \omega}^{\rm in}$ has no outcoming wave from past horizon, while $R^{\rm up}$ has no incoming wave at past infinity. Since these are the property of waves causally generated by a source, a solution of the Teukolsky equation which has purely outgoing property at infinity and has purely ingoing property at the horizon is given by

$$R_{\ell m \omega} = \frac{1}{W_{\ell m \omega}} \left\{ R_{\ell m \omega}^{\rm up} \int_{r_+}^{r} dr' R_{\ell m \omega}^{\rm in} T_{\ell m \omega} \Delta^{-2} + R_{\ell m \omega}^{\rm in} \int_{r}^{\infty} dr' R_{\ell m \omega}^{\rm up} T_{\ell m \omega} \Delta^{-2} \right\}, \tag{17}$$

where the Wronskian $W_{\ell m \omega}$ is given by

$$W_{\ell m \omega} = 2i\omega C_{\ell m \omega}^{\rm trans} B_{\ell m \omega}^{\rm inc}. \tag{18}$$

Then, the asymptotic behavior at the horizon is

$$R_{\ell m \omega}(r \to r_+) \to \frac{B_{\ell m \omega}^{\rm trans} \Delta^2 e^{-ikr^*}}{2i\omega C_{\ell m \omega}^{\rm trans} B_{\ell m \omega}^{\rm inc}} \int_{r_+}^{\infty} dr' R_{\ell m \omega}^{\rm up} T_{\ell m \omega} \Delta^{-2} \equiv \tilde{Z}_{\ell m \omega}^{\rm H} \Delta^2 e^{-ikr^*}, \tag{19}$$

while the asymptotic behavior at infinity is

$$R_{\ell m \omega}(r \to \infty) \to \frac{r^3 e^{i\omega r^*}}{2i\omega B_{\ell m \omega}^{\rm inc}} \int_{r_+}^{\infty} dr' \frac{T_{\ell m \omega}(r') R_{\ell m \omega}^{\rm in}(r')}{\Delta^2(r')} \equiv \tilde{Z}_{\ell m \omega}^{\infty} r^3 e^{i\omega r^*}. \tag{20}$$

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We note that the homogeneous Teukolsky equation is invariant under the complex conjugation followed by the transformation $m \to -m$ and $\omega \to -\omega$. Thus, we can set $\bar{R}^{\text{in,up}}_{\ell m \omega} = R^{\text{in,up}}_{\ell -m -\omega}$, where the bar denote the complex conjugation.

We consider $T_{\mu\nu}$ of a monopole particle of mass μ. The energy momentum tensor takes the form,

$$T^{\mu\nu} = \frac{\mu}{\Sigma \sin\theta dt/d\tau} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)). \qquad (21)$$

where $z^\mu = \big(t, r(t), \theta(t), \varphi(t)\big)$ is a geodesic trajectory and $\tau = \tau(t)$ is the proper time along the geodesic. The geodesic equations in the Kerr geometry are given by

$$\Sigma \frac{d\theta}{d\tau} = \pm \Big[C - \cos^2\theta \Big\{ a^2(1 - E^2) + \frac{l_z^2}{\sin^2\theta} \Big\} \Big]^{1/2} \equiv \Theta(\theta),$$

$$\Sigma \frac{d\varphi}{d\tau} = -\Big(aE - \frac{l_z}{\sin^2\theta} \Big) + \frac{a}{\Delta}\Big(E(r^2 + a^2) - al_z \Big) \equiv \Phi,$$

$$\Sigma \frac{dt}{d\tau} = -\Big(aE - \frac{l_z}{\sin^2\theta} \Big) a\sin^2\theta + \frac{r^2 + a^2}{\Delta}\Big(E(r^2 + a^2) - al_z \Big) \equiv T,$$

$$\Sigma \frac{dr}{d\tau} = \pm\sqrt{R}, \qquad (22)$$

where E, l_z and C are the energy, the z-component of the angular momentum and the Carter constant of a test particle, respectively,[1] and

$$R = [E(r^2 + a^2) - al_z]^2 - \Delta[(Ea - l_z)^2 + r^2 + C]. \qquad (23)$$

Using Eq. (22), the tetrad components of the energy momentum tensor are expressed as

$$T_{nn} = \mu \frac{C_{nn}}{\sin\theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)),$$

$$T_{\overline{m}n} = \mu \frac{C_{\overline{m}n}}{\sin\theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)), \qquad (24)$$

$$T_{\overline{m}\,\overline{m}} = \mu \frac{C_{\overline{m}\,\overline{m}}}{\sin\theta} \delta(r - r(t))\delta(\theta - \theta(t))\delta(\varphi - \varphi(t)),$$

where

$$C_{nn} = \frac{1}{4\Sigma^3 \dot{t}} \Big[E(r^2 + a^2) - al_z + \Sigma \frac{dr}{d\tau} \Big]^2,$$

---

[1]These constants of motion are those measured in units of μ. That is, if expressed in the standard units, E, l_z and C in Eq. (22) are to be replaced with E/μ, l_z/μ and C/μ^2, respectively.

$$
C_{\overline{m}\,n} = -\frac{\rho}{2\sqrt{2}\Sigma^2\dot{t}}\left[E(r^2+a^2)-al_z+\Sigma\frac{dr}{d\tau}\right]\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)+\Sigma\frac{d\theta}{d\tau}\right],
$$

$$
C_{\overline{m}\,\overline{m}} = \frac{\rho^2}{2\Sigma\dot{t}}\left[i\sin\theta\left(aE-\frac{l_z}{\sin^2\theta}\right)+\Sigma\frac{d\theta}{d\tau}\right]^2, \tag{25}
$$

and $\dot{t}=dt/d\tau$. Substituting Eq. (6) into Eq. (13) and performing integration by part, we obtain

$$
\begin{aligned}
T_{\ell m\omega} =\ & \frac{4\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\int d\theta e^{i\omega t-im\varphi(t)} \\
& \times\Bigg[-\frac{1}{2}L_1^{\dagger}\{\rho^{-4}L_2^{\dagger}(\rho^3 S)\}C_{n\,n}\rho^{-2}\overline{\rho}^{-1}\delta(r-r(t))\delta(\theta-\theta(t)) \\
& +\frac{\Delta^2\overline{\rho}^2}{\sqrt{2}\rho}(L_2^{\dagger}S+ia(\overline{\rho}-\rho)\sin\theta S)J_+\left\{\frac{C_{\overline{m}\,n}}{\rho^2\overline{\rho}^2\Delta}\delta(r-r(t))\delta(\theta-\theta(t))\right\} \\
& +\frac{1}{2\sqrt{2}}L_2^{\dagger}\{\rho^3 S(\overline{\rho}^2\rho^{-4})_{,r}\}C_{\overline{m}\,n}\Delta\rho^{-2}\overline{\rho}^{-2}\delta(r-r(t))\delta(\theta-\theta(t)) \\
& -\frac{1}{4}\rho^3\Delta^2 SJ_+\{\rho^{-4}J_+(\overline{\rho}\rho^{-2}C_{\overline{m}\,\overline{m}}\delta(r-r(t))\delta(\theta-\theta(t)))\}\Bigg], \tag{26}
\end{aligned}
$$

where

$$
L_s^{\dagger}=\partial_\theta-\frac{m}{\sin\theta}+a\omega\sin\theta+s\cot\theta, \tag{27}
$$

and S denotes $_{-2}S_{\ell m}^{a\omega}(\theta)$ for simplicity.

For a source bounded in a finite range of r, it is convenient to rewrite Eq. (26) further as

$$
\begin{aligned}
T_{\ell m\omega} =\ & \mu\int_{-\infty}^{\infty}dt e^{i\omega t-im\varphi(t)}\Delta^2\Big[(A_{n\,n\,0}+A_{\overline{m}\,n\,0}+A_{\overline{m}\,\overline{m}\,0})\delta(r-r(t)) \\
& +\ \{(A_{\overline{m}\,n\,1}+A_{\overline{m}\,\overline{m}\,1})\delta(r-r(t))\}_{,r}+\{A_{\overline{m}\,\overline{m}\,2}\delta(r-r(t))\}_{,rr}\Big], \tag{28}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{n\,n\,0} &= \frac{-2}{\sqrt{2\pi}\Delta^2}C_{n\,n}\rho^{-2}\overline{\rho}^{-1}L_1^+\{\rho^{-4}L_2^+(\rho^3 S)\}, \\
A_{\overline{m}\,n\,0} &= \frac{2}{\sqrt{\pi}\Delta}C_{\overline{m}\,n}\rho^{-3}\Big[(L_2^+ S)\left(\frac{iK}{\Delta}+\rho+\overline{\rho}\right) \\
&\qquad -a\sin\theta S\frac{K}{\Delta}(\overline{\rho}-\rho)\Big], \\
A_{\overline{m}\,\overline{m}\,0} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S\Big[-i\left(\frac{K}{\Delta}\right)_{,r}-\frac{K^2}{\Delta^2}+2i\rho\frac{K}{\Delta}\Big], \\
A_{\overline{m}\,n\,1} &= \frac{2}{\sqrt{\pi}\Delta}\rho^{-3}C_{\overline{m}\,n}[L_2^+ S+ia\sin\theta(\overline{\rho}-\rho)S], \\
A_{\overline{m}\,\overline{m}\,1} &= -\frac{2}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S\left(i\frac{K}{\Delta}+\rho\right), \\
A_{\overline{m}\,\overline{m}\,2} &= -\frac{1}{\sqrt{2\pi}}\rho^{-3}\overline{\rho}C_{\overline{m}\,\overline{m}}S. \tag{29}
\end{aligned}
$$

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Inserting Eq. (28) into Eq. (20), we obtain $\tilde{Z}_{\ell m\omega}$ as

$$\tilde{Z}_{\ell m\omega} = \frac{\mu}{2i\omega B_{\ell m\omega}^{\rm inc}} \int_{-\infty}^{\infty} dt\, e^{i\omega t - im\varphi(t)} W_{\ell m\omega}\,, \tag{30}$$

where

$$\begin{aligned}
W_{\ell m\omega} &= \Big[R_{\ell m\omega}^{\rm in}\{A_{n\,n\,0} + A_{\overline{m}\,n\,0} + A_{\overline{m}\,\overline{m}\,0}\} \\
&\quad - \frac{dR_{\ell m\omega}^{\rm in}}{dr}\{A_{\overline{m}\,n\,1} + A_{\overline{m}\,\overline{m}\,1}\} + \frac{d^2 R_{\ell m\omega}^{\rm in}}{dr^2} A_{\overline{m}\,\overline{m}\,2} \Big]_{r=r(t)}. \tag{31}
\end{aligned}$$

In this paper, we focus on orbits which are either circular (with or without inclination) or eccentric but confined on the equatorial plane. In either case, the frequency spectrum of $T_{\ell m\omega}$ becomes discrete. Accordingly, $\tilde{Z}_{\ell m\omega}$ in Eq. (19) or (20) takes the form,

$$\tilde{Z}_{\ell m\omega} = \sum_n \delta(\omega - \omega_n) Z_{\ell m\omega}\,. \tag{32}$$

Then, in particular, ψ_4 at $r \to \infty$ is obtained from Eq. (8) as

$$\psi_4 = \frac{1}{r} \sum_{\ell m n} Z_{\ell m\omega_n} \frac{{}_{-2}S_{\ell m}^{a\omega_n}}{\sqrt{2\pi}} e^{i\omega_n(r^*-t)+im\varphi}. \tag{33}$$

At infinity, ψ_4 is related to the two independent modes of gravitational waves h_+ and $h_\times$ as

$$\psi_4 = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times). \tag{34}$$

From Eqs. (33) and (34), the luminosity averaged over $t \gg \Delta t$, where Δt is the characteristic time scale of the orbital motion (e.g., a period between the two consecutive apastrons), is given by

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{\ell,m,n} \frac{\left| Z_{\ell m\omega_n} \right|^2}{4\pi\omega_n^2} \equiv \sum_{\ell,m,n} \left(\frac{dE}{dt} \right)_{\ell m n}. \tag{35}$$

In the same way, the time-averaged angular momentum flux is given by

$$\left\langle \frac{dJ_z}{dt} \right\rangle = \sum_{\ell,m,n} \frac{m \left| Z_{\ell m\omega_n} \right|^2}{4\pi\omega_n^3} \equiv \sum_{\ell,m,n} \left(\frac{dJ_z}{dt} \right)_{\ell m n} = \sum_{\ell,m,n} \frac{m}{\omega_n} \left(\frac{dE}{dt} \right)_{\ell m n}. \tag{36}$$

2.2 Chandrasekhar-Sasaki-Nakamura transformation

As seen from the asymptotic behaviors of the radial functions given in Eqs. (19) and (20), the Teukolsky equation is not in the form of a canonical wave equation near the horizon and infinity. Therefore, it is desirable to find a transformation

that brings the radial Teukolsky equation into the form of a standard wave equation.

In the Schwarzschild case, Chandrasekhar found that the Teukolsky equation can be transformed to the Regge-Wheeler equation, which has the standard form of a wave equation with solutions having regular asymptotic behaviors at horizon and infinity [19]. The Regge-Wheeler equation was originally derived as an equation governing the odd parity metric perturbation [61]. The existence of this transformation implies that the Regge-Wheeler equation can describe the even parity metric perturbation simultaneously, though the explicit relation of the Regge-Wheeler function obtained by the Chandrasekhar transformation with the actual metric perturbation variables has not been given in the literature yet.

Later, Sasaki and Nakamura succeeded in generalizing the Chandrasekhar transformation to the Kerr case [66, 67]. The Chandrasekhar-Sasaki-Nakamura transformation was originally introduced to make the potential in the radial equation short-ranged and to make the source term well-behaved at horizon and infinity. Since we are interested only in bound orbits, it is not necessary to perform this transformation. Nevertheless, because its flat space limit reduces to the standard radial wave equation in the Minkowski spacetime, it is convenient to apply the transformation when dealing with the post-Minkowski or post-Newtonian expansion, at least at low orders of expansion.

We transform the homogeneous Teukolsky equation to the Sasaki-Nakamura equation [66, 67], which is given by

$$\left[\frac{d^2}{dr^{*2}} - F(r)\frac{d}{dr^*} - U(r)\right]X_{\ell m\omega} = 0. \tag{37}$$

The function $F(r)$ is given by

$$F(r) = \frac{\eta_{,r}}{\eta}\frac{\Delta}{r^2 + a^2}, \tag{38}$$

where

$$\eta = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4, \tag{39}$$

with

$$
\begin{aligned}
c_0 &= -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m), \\
c_1 &= 8ia[3a\omega - \lambda(a\omega - m)], \\
c_2 &= -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \\
c_3 &= 24ia^3(a\omega - m) - 24Ma^2, \\
c_4 &= 12a^4.
\end{aligned}
\tag{40}
$$

The function $U(r)$ is given by

$$U(r) = \frac{\Delta U_1}{(r^2 + a^2)^2} + G^2 + \frac{\Delta G_{,r}}{r^2 + a^2} - FG, \tag{41}$$

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where

$$G = -\frac{2(r-M)}{r^2+a^2} + \frac{r\Delta}{(r^2+a^2)^2},$$

$$U_1 = V + \frac{\Delta^2}{\beta}\Big[\Big(2\alpha + \frac{\beta_{,r}}{\Delta}\Big)_{,r} - \frac{\eta_{,r}}{\eta}\Big(\alpha + \frac{\beta_{,r}}{\Delta}\Big)\Big],$$

$$\alpha = -i\frac{K\beta}{\Delta^2} + 3iK_{,r} + \lambda + \frac{6\Delta}{r^2},$$

$$\beta = 2\Delta\Big(-iK + r - M - \frac{2\Delta}{r}\Big). \tag{42}$$

The relation between $R_{\ell m\omega}$ and $X_{\ell m\omega}$ is

$$R_{\ell m\omega} = \frac{1}{\eta}\Big\{\Big(\alpha + \frac{\beta_{,r}}{\Delta}\Big)\chi_{\ell m\omega} - \frac{\beta}{\Delta}\chi_{\ell m\omega,r}\Big\}, \tag{43}$$

where $\chi_{\ell m\omega} = X_{\ell m\omega}\Delta/(r^2+a^2)^{1/2}$. Conversely, we can express $X_{\ell m\omega}$ in terms of $R_{\ell m\omega}$ as

$$X_{\ell m\omega} = (r^2+a^2)^{1/2}\, r^2\, J_- J_- \left[\frac{1}{r^2}R_{\ell m\omega}\right], \tag{44}$$

where $J_- = (d/dr) - i(K/\Delta)$.

If we set $a = 0$, this transformation reduces to the Chandrasekhar transformation for the Schwarzschild black hole [19]. The explicit form of the transformation is

$$R_{\ell m\omega} = \frac{\Delta}{c_0}\Big(\frac{d}{dr^*} + i\omega\Big)\frac{r^2}{\Delta}\Big(\frac{d}{dr^*} + i\omega\Big)(rX_{\ell m\omega}), \tag{45}$$

$$X_{\ell m\omega} = \frac{r^5}{\Delta}\Big(\frac{d}{dr^*} - i\omega\Big)\frac{r^2}{\Delta}\Big(\frac{d}{dr^*} + i\omega\Big)\frac{R_{\ell m\omega}}{r^2}, \tag{46}$$

where c_0, defined in Eq. (40), reduces to $c_0 = (\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega$. In this case, the Sasaki-Nakamura equation (37) reduces to the Regge-Wheeler equation [61] which is given by

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\right]X_{\ell\omega}(r) = 0, \tag{47}$$

where

$$V(r) = \Big(1 - \frac{2M}{r}\Big)\Big(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\Big). \tag{48}$$

As clear from the above form of the equation, the lowest order solutions are given by the spherical Bessel functions. Hence it is intuitively straightforward to apply the post-Newtonian expansion to it. Some useful techniques for the post-Newtonian expansion were developed for the Schwarzschild case by Poisson [56] and Sasaki [65].

The asymptotic behavior of the ingoing-wave solution $X^{\rm in}$ which corresponds to Eq. (14) is

$$X_{\ell m\omega}^{\rm in} \to \begin{cases} A_{\ell m\omega}^{\rm ref}e^{i\omega r^*} + A_{\ell m\omega}^{\rm inc}e^{-i\omega r^*} & \text{for } r^* \to \infty \\ A_{\ell m\omega}^{\rm trans}e^{-ikr^*} & \text{for } r^* \to -\infty. \end{cases} \tag{49}$$

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The coefficients A^{inc}, A^{ref} and A^{trans} are related to B^{inc}, B^{ref} and B^{trans}, defined in Eq. (14), by

$$B^{\text{inc}}_{\ell m\omega} \;=\; -\frac{1}{4\omega^2}A^{\text{inc}}_{\ell m\omega}, \tag{50}$$

$$B^{\text{ref}}_{\ell m\omega} \;=\; -\frac{4\omega^2}{c_0}A^{\text{ref}}_{\ell m\omega}, \tag{51}$$

$$B^{\text{trans}}_{\ell m\omega} \;=\; \frac{1}{d_{\ell m\omega}}A^{\text{trans}}_{\ell m\omega}, \tag{52}$$

where

$$
\begin{aligned}
d_{\ell m\omega} \;=\; & \sqrt{2Mr_+}[(8 - 24iM\omega - 16M^2\omega^2)r_+^2 \\
& + (12iam - 16M + 16amM\omega + 24iM^2\omega)r_+ \\
& - 4a^2m^2 - 12iamM + 8M^2].
\end{aligned}
$$

In the following sections, we present a method of post-Newtonian expansion based on the above formalism in the case of the Schwarzschild background. In the Kerr case, although a post-Newtonian expansion method developed in previous work [69, 74] was based on the Sasaki-Nakamura equation, we will not present it in this paper. Instead, we present a different formalism, namely the one developed by Mano, Suzuki and Takasugi which allows us to solve the Teukolsky equation in a more systematic manner, albeit very mathematical [43]. The reason is that the equations in the Kerr case are already complicated enough even if one uses the Sasaki-Nakamura equation, so that there is not much advantage in using it. In contrast, in the Schwarzschild case, it is much easier to obtain physical insight into the role of relativistic corrections if we deal with the Regge-Wheeler equation.

3 Post-Newtonian expansion of the Regge-Wheeler equation

In this section, we review a post-Newtonian expansion method for the Schwarzschild background, based on the Regge-Wheeler equation. We focus on the gravitational waves emitted to infinity, but not on those absorbed by the black hole. The black hole absorption is deferred to Section 4, in which we review the Mano-Suzuki-Takasugi method for solving the Teukolsky equation.

Since we are interested in the waves emitted to infinity, as seen from Eq. (20), what we need is a method to evaluate the ingoing-wave Teukolsky function $R^{\text{in}}_{\ell m\omega}$, or its counterpart of the Regge-Wheeler equation, $X^{\text{in}}_{\ell m\omega}$, which are related by Eq. (43). In addition, we assume $\omega > 0$ whenever it is necessary throughout this section. Formulas and equations for $\omega < 0$ are obtained from the symmetry $\bar{X}^{\text{in}}_{\ell m\omega} = X^{\text{in}}_{\ell -m-\omega}$.

3.1 Basic assumptions

We consider the case of a test particle with mass μ in an orbit which is nearly circular around a black hole with mass $M \gg \mu$. For a nearly circular orbit, say at $r \sim r_0$, what we need to know is the behavior of $R^{\text{in}}_{\ell m \omega}$ at $r \sim r_0$. In addition, the contribution of ω to $R^{\text{in}}_{\ell m \omega}$ comes mainly from $\omega \sim m\Omega_\varphi$, where $\Omega_\varphi \sim (M/r_0^3)^{1/2}$ is the orbital angular frequency.

Thus, if we express the Regge-Wheeler equation (47) in terms of a non-dimensional variable $z \equiv \omega r$, with a non-dimensional parameter $\epsilon \equiv 2M\omega$, we are interested in the behavior of $X^{\text{in}}_{\ell m \omega}(z)$ at $z \sim \omega r_0 \sim m(M/r_0)^{1/2} \sim v$ with $\epsilon \sim 2m(M/r_0)^{3/2} \sim v^3$, where $v \equiv (M/r_0)^{1/2}$ is the characteristic orbital velocity. The post-Newtonian expansion assumes that v is much smaller than the velocity of light; $v \ll 1$. Consequently, we have $\epsilon \ll v \ll 1$ under the post-Newtonian expansion.

To obtain $X^{\text{in}}_{\ell m \omega}$ (which we denote below by X_ℓ for simplicity) under these assumptions, we find it convenient to rewrite the Regge-Wheeler equation in an alternative form. It is

$$\left[\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)\right]\xi_\ell = \epsilon e^{-iz} \frac{d}{dz}\left[\frac{1}{z^3}\frac{d}{dz}\left(e^{iz}z^2\xi_\ell(z)\right)\right], \qquad (53)$$

where ξ_ℓ is a function related to X_ℓ as

$$X_\ell = z e^{-i\epsilon \ln(z-\epsilon)}\xi_\ell. \qquad (54)$$

The ingoing-wave boundary condition of ξ_ℓ is derived from Eqs. (49) and (54) as

$$\xi_\ell \rightarrow \begin{cases} A^{\text{inc}}_\ell e^{i\epsilon \ln \epsilon} z^{-1} e^{-iz} + A^{\text{ref}}_\ell e^{-i\epsilon \ln \epsilon} z^{-1} e^{i(z+2\epsilon \ln z)} & \text{for } r^* \rightarrow \infty, \\ A^{\text{trans}}_\ell \epsilon^{-1} e^{i\epsilon(\ln \epsilon - 1)} & \text{for } r^* \rightarrow -\infty. \end{cases} \qquad (55)$$

The above form of the Regge-Wheeler equation is used in the following subsections.

It should be noted that if we recover the gravitational constant G, we have $\epsilon = 2GM\omega$. Thus, the expansion in terms of ϵ corresponds to the post-Minkowski expansion, and expanding the Regge-Wheeler equation with the assumption $\epsilon \ll 1$ gives a set of iterative wave equations on the flat spacetime background. One of the most significant differences between the black hole perturbation theory and any theory based on the flat spacetime background is the presence of the black hole horizon in the former case. Thus, if we naively expand the Regge-Wheeler equation with respect to ϵ, the horizon boundary condition becomes unclear, since there is no horizon on the flat spacetime. To establish the boundary condition at the horizon, we need to treat the Regge-Wheeler equation near the horizon separately. We thus have to find a solution near the horizon and the solution obtained by the post-Minkowski expansion must be matched with it in the region where both solutions are valid.

It may be of interest to note the difference between the matching used in the BDI approach for the post-Newtonian expansion [12, 15] and the matching used

here. In the BDI approach, the matching is done between the post-Minkowskian metric and the near-zone post-Newtonian metric. In our case, the matching is done between the post-Minkowskian gravitational field and the gravitational field near the black hole horizon.

3.2   Horizon solution; $z \ll 1$

In this subsection, we first consider the solution near the horizon, which we call the horizon solution, based on [59]. To do so, we assume $z \ll 1$ and treat ϵ as a small number, but leave the ratio z/ϵ arbitrary. We change the independent variable to $x = 1 - z/\epsilon$ and the wave function to

$$Z = \left(\frac{\epsilon}{z}\right)^3 \frac{X_\ell}{A_\ell^{\text{trans}}} = \left(\frac{\epsilon}{z}\right)^2 \frac{\epsilon\,\xi_\ell}{A_\ell^{\text{trans}} e^{i\epsilon(\ln\epsilon - 1)}} \,. \tag{56}$$

Note that the horizon corresponds to $x = 0$. We then have

$$x(x-1)Z'' + \big[2(3-i\epsilon)x - (1-2i\epsilon)\big]Z'$$
$$+ \big[6 - \ell(\ell+1) - 5i\epsilon + \epsilon^2(2 - 3x + x^2)\big]Z = 0, \tag{57}$$

where a prime denotes differentiation with respect to x. We look for a solution which is regular at $x = 0$.

First, we consider the lowest order solution by setting $\epsilon = 0$ in Eq. (57). The boundary condition (55) requires that $Z = 1$ at $x = 0$. The solution which satisfies the boundary condition is

$$Z = \sum_{n=0}^{\ell-2} \frac{(2-\ell)_n(\ell+3)_n}{n!}x^n \,; \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{58}$$

Thus, the lowest order solution is a polynomial of order $\ell - 2$ in $x = 1 - z/\epsilon$.

Next, we consider the solution accurate to $O(\epsilon)$. We neglect the terms of $O(\epsilon^2)$ in Eq. (57). Then its takes the form of a hypergeometric equation,

$$x(x-1)Z'' + \big[(a+b+1)x - c\big]Z' + abZ = 0, \tag{59}$$

with parameters

$$\begin{aligned} a &= -(\ell-2) - i\epsilon + O(\epsilon^2), \\ b &= \ell + 3 - i\epsilon + O(\epsilon^2), \\ c &= 1 - 2i\epsilon. \end{aligned} \tag{60}$$

The two linearly independent solutions are $F(a, b; c; x)$ and $x^{1-c}F(a+1-c, b+1-c; 2-c; x)$, where F is the hypergeometric function. However, only the first solution is regular at $x = 0$. We therefore obtain

$$\xi_\ell(z) = A_\ell^{\text{trans}} \epsilon^{-1} e^{i\epsilon(\ln\epsilon - 1)} \left(\frac{z}{\epsilon}\right)^2 F\left(a, b; c; 1 - \frac{z}{\epsilon}\right). \tag{61}$$

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The above solution must be matched with the solution obtained from the post-Minkowski expansion of Eq. (53), which we call the outer solution, in a region both solutions are valid. It is the region where the post-Newtonian expansion is applied, i.e., the region $\epsilon \ll z \ll 1$. For this purpose, we rewrite Eq. (61) as (see, e.g., Eq. (15.3.8) of [7])

$$
\begin{aligned}
\xi_\ell &= A_\ell^{\text{trans}} \epsilon^{-1} e^{i\epsilon(\ln \epsilon - 1)} \left[\left(\frac{z}{\epsilon} \right)^{\ell + i\epsilon} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} F(a, c-b; a-b+1; \frac{\epsilon}{z}) \right. \\
&\left. + \left(\frac{z}{\epsilon} \right)^{-\ell - 1 + i\epsilon} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F(b, c-a; b-a+1; \frac{\epsilon}{z}) \right].
\end{aligned}
\tag{62}
$$

This naturally allows the expansion in ϵ/z. It should be noted that the second term in the square brackets of the above expression is meaningless as it is, since the factor $\Gamma(a - b)$ diverges for integer ℓ. So, when evaluating the second term, we first have to extend ℓ to a non-integer number. Then, only after expanding it in terms of ϵ, we should take the limit of an integer ℓ. One then finds that this procedure gives rise to an additional factor of $O(\epsilon)$. For $\epsilon/z \ll 1$, it therefore becomes $O(\epsilon^{2\ell+2})$ higher in ϵ than the first term. Then we obtain

$$
\begin{aligned}
\xi_\ell(\epsilon \ll z \ll 1) &= \frac{(2\ell)!}{(\ell-2)!(\ell+2)!} \frac{z^\ell}{\epsilon^{\ell+1}} \left[1 + i\epsilon(a_\ell + \ln z) \right. \\
&\left. - \frac{(\ell-2)(\ell+2)}{2\ell} \frac{\epsilon}{z} + O(\epsilon^2) \right],
\end{aligned}
\tag{63}
$$

where

$$
a_\ell = 2\gamma + \psi(\ell - 1) + \psi(\ell + 3) - 1,
\tag{64}
$$

and $\psi(n)$ is the digamma function,

$$
\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1},
\tag{65}
$$

and $\gamma \simeq 0.57721$ is the Euler constant.

As we will see below, the above solution is accurate enough to determine the boundary condition of the outer solution up to 6PN order of expansion.

3.3 Outer solution; $\epsilon \ll 1$

We now solve Eq. (53) in the limit $\epsilon \ll 1$, i.e., by applying the post-Minkowski expansion to it. In this subsection, we consider the solution to $O(\epsilon)$. Then we match the solution to the horizon solution given by Eq. (63) at $\epsilon \ll z \ll 1$.

By setting

$$
\xi_\ell(z) = \sum_{n=0}^\infty \epsilon^n \xi_\ell^{(n)}(z),
\tag{66}
$$

each $\xi_\ell^{(n)}(z)$ is found to satisfy

$$
\left[\frac{d^2}{dz^2} + \frac{2}{z} \frac{d}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2} \right) \right] \xi_\ell^{(n)} = e^{-iz} \frac{d}{dz} \left[\frac{1}{z^3} \frac{d}{dz} \left(e^{iz} z^2 \xi_\ell^{(n-1)}(z) \right) \right].
\tag{67}
$$

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Equation (67) is an inhomogeneous spherical Bessel equation. It is the simplicity of this equation which motivated the introduction of the auxiliary function ξ_ℓ [65].

The zeroth-order solution, $\xi_\ell^{(0)}$, satisfies the homogeneous spherical Bessel equation, and must be a linear combination of the spherical Bessel functions of the first and second kinds, $j_\ell(z)$ and $n_\ell(z)$. Here, we demand the compatibility with the horizon solution (63). Since $j_\ell(z) \sim z^\ell$ and $n_\ell(z) \sim z^{-\ell-1}$, $n_\ell(z)$ does not match with the horizon solution at the leading order of ϵ. We therefore have

$$\xi_\ell^{(0)}(z) = \alpha_\ell^{(0)} j_\ell(z). \tag{68}$$

The constant $\alpha_\ell^{(0)}$ represents the overall normalization of the solution. Since it can be chosen arbitrary, we set $\alpha_\ell^{(0)} = 1$ below.

The procedure to obtain $\xi_\ell^{(1)}(z)$ was described in detail in [65]. Using the Green function $G(z, z') = j_\ell(z_<) n_\ell(z_>)$, Eq. (67) may be put into an indefinite integral form,

$$
\begin{aligned}
\xi_\ell^{(n)} \;=\; & n_\ell \int^z dz z^2 e^{-iz} j_\ell \left[\frac{1}{z^3} (e^{iz} z^2 \xi_\ell^{(n-1)}(z))' \right]' \\
& - j_\ell \int^z dz z^2 e^{-iz} n_\ell \left[\frac{1}{z^3} (e^{iz} z^2 \xi_\ell^{(n-1)}(z))' \right]'.
\end{aligned}
\tag{69}
$$

The calculation is tedious but straightforward. All the necessary formulas to obtain $\xi_\ell^{(n)}$ for $n \leq 2$ are given in Appendix of [65] or Appendix D of [46]. Using those formulas, we have for $n = 1$,

$$
\begin{aligned}
\xi_\ell^{(1)} \;=\; & \alpha_\ell^{(1)} j_\ell + \beta_\ell^{(1)} n_\ell \\
& + \frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)} j_{\ell+1} - \left(\frac{\ell^2 - 4}{2\ell(2\ell+1)} + \frac{2\ell - 1}{\ell(\ell-1)} \right) j_{\ell-1} \\
& + R_{\ell,0} j_0 + \sum_{m=1}^{\ell-2} \left(\frac{1}{m} + \frac{1}{m+1} \right) R_{\ell,m} j_m - 2 D_\ell^{nj} + i j_\ell \ln z.
\end{aligned}
\tag{70}
$$

Here, D_ℓ^{nj} and $R_{\ell,m}$ are functions defined as follows. The function D_ℓ^{nj} is given by

$$D_\ell^{nj} = \frac{1}{2} \left[j_\ell \, \mathrm{Si}(2z) - n_\ell \left(\mathrm{Ci}(2z) - \gamma - \ln 2z \right) \right], \tag{71}$$

where $\mathrm{Ci}(x) = -\int_x^\infty dt \cos t / t$ and $\mathrm{Si}(x) = \int_0^x dt \sin t / t$. The function $R_{m,k}$ is defined by $R_{m,k} = z^2 (n_m j_k - j_m n_k)$. It is a polynomial in inverse powers of z given by

$$R_{m,k} = - \sum_{r=0}^{[(m-k-1)/2]} (-1)^r \frac{\Gamma(m - k - r)\Gamma\left(m + \frac{1}{2} - r\right)}{r! \, \Gamma(m - k - 2r)\Gamma\left(k + \frac{3}{2} + r\right)} \left(\frac{2}{z} \right)^{m-k-1-2r}, \tag{72}$$

for $m > k$, and

$$R_{m,k} = -R_{k,m}. \tag{73}$$

for $m < k$.

Here, we again perform the matching with the horizon solution (63). It should be noted that $\xi_\ell^{(1)}$ given by Eq. (70) is regular in the limit $z \to 0$ except for the term $\beta_\ell^{(1)} n_\ell$. By examining the asymptotic behavior of Eq. (70) at $z \ll 1$, we find $\beta_\ell^{(1)} = 0$, i.e., the solution is regular at $z = 0$. As for $\alpha_\ell^{(1)}$, it only contributes to the renormalization of $\alpha_\ell^{(0)}$. Hence we set $\alpha_\ell^{(1)} = 0$. Thus, the transmission amplitude A_ℓ^{trans} is determined to $O(\epsilon)$ as

$$A_\ell^{\text{trans}} = \frac{(\ell-2)!(\ell+2)!}{(2\ell)!(2\ell+1)!} \epsilon^{\ell+1}[1 - i\epsilon\, a_\ell + O(\epsilon^2)] . \tag{74}$$

It may be noted that this explicit expression for A_ℓ^{trans} is unnecessary for the evaluation of gravitational waves at infinity. It is relevant only for the evaluation of the black hole absorption.

3.4 More on the inner boundary condition of the outer solution

In this subsection, we discuss the inner boundary condition of the outer solution in more detail. As we have seen in the previous subsection, the boundary condition of ξ_ℓ is that it is regular at $z \to 0$, at least to $O(\epsilon)$. While in the full non-linear level, the horizon boundary is at $z = \epsilon$. We therefore investigate to what order in ϵ the condition of regularity at $z = 0$ can be applied.

Let us consider the general form of the horizon solution. With $x = 1 - z/\epsilon$, it is expanded in the form,

$$\xi_\ell = \xi_\ell^{\{0\}}(x) + \epsilon\, \xi_\ell^{\{1\}}(x) + \epsilon^2\, \xi_\ell^{\{2\}}(x) + \cdots . \tag{75}$$

The lowest order solution $\xi_\ell^{\{0\}}(x)$ is given by the polynomial (58). Apart from the common overall factor, it is schematically expressed as

$$\xi_\ell^{\{0\}} = \left(\frac{z}{\epsilon}\right)^\ell \left[1 + c_1 \frac{\epsilon}{z} + \cdots + c_{\ell-2} \left(\frac{\epsilon}{z}\right)^{\ell-2}\right] . \tag{76}$$

Thus, $\xi_\ell^{\{0\}}$ does not have a term matched with n_ℓ, but it matches with j_ℓ. We have $\xi_\ell^{\{0\}} = z^\ell \epsilon^{-\ell} \sim \epsilon^{-\ell} j_\ell$. A term that matches with n_ℓ first appears from $\xi_\ell^{\{1\}}$. This can be seen from the horizon solution valid to $O(\epsilon)$, Eq. (62). The second term in the square brackets of it produces a term $\epsilon\,(z/\epsilon)^{-\ell-1} = \epsilon^{\ell+2} z^{-\ell-1} \sim \epsilon^{\ell+2} n_\ell$. This term therefore becomes $O(\epsilon^{2\ell+2}/z^{2\ell+1})$ higher than the lowest order term $\epsilon^{-\ell} j_\ell$. Since $\ell \geq 2$, this effect first appears at $O(\epsilon^6)$ in the post-Minkowski expansion, while it first appears at $O(v^{13})$ in the post-Newtonian expansion if we note that $\epsilon = O(v^3)$ and $z = O(v)$. This implies, in particular, that if we are interested in the gravitational waves emitted to infinity, we may simply impose the regularity at $z = 0$ as the inner boundary condition of the outer solution for the calculation up to 6PN order beyond the quadrupole formula.

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The above fact that a non-trivial boundary condition due to the presence of the black hole horizon appears at $O(\epsilon^{2\ell+2})$ in the post-Minkowski expansion can be more easily seen as follows. Since $j_\ell = O(z^\ell)$ as $z \to 0$, we have $X_\ell \to O(\epsilon^{\ell+1})e^{-iz^*}$, or $A_\ell^{\text{trans}} = O(\epsilon^{\ell+1})$, where $z^* = z + \epsilon \ln(z - \epsilon)$. On the other hand, from the asymptotic behavior of j_ℓ at $z = \infty$, the coefficients A_ℓ^{inc} and A_ℓ^{ref} must be of order unity. Then using the Wronskian argument, we find

$$|A_\ell^{\text{inc}}| - |A_\ell^{\text{ref}}| = \frac{|A_\ell^{\text{trans}}|^2}{|A_\ell^{\text{inc}}| + |A_\ell^{\text{ref}}|} = O(\epsilon^{2\ell+2}). \tag{77}$$

Thus, we immediately see that a non-trivial boundary condition appears at $O(\epsilon^{2\ell+2})$.

It is also useful to keep in mind the above fact when we solve for ξ_ℓ under the post-Minkowski expansion. It implies that we may choose a phase so that A_ℓ^{inc} and A_ℓ^{ref} are complex conjugate to each other, to $O(\epsilon^{2\ell+1})$. With this choice, the imaginary part of X_ℓ, which reflects the boundary condition at horizon, do not appear until $O(\epsilon^{2\ell+2})$ because the Regge-Wheeler equation is real. Then recalling the relation of ξ_ℓ to X_ℓ, Eq. (54), $\text{Im}\left(\xi_\ell^{(n)}\right)$ for a given $n \leq 2\ell + 1$ is completely determined in terms of $\text{Re}\left(\xi_\ell^{(r)}\right)$ for $r \leq n - 1$. That is, we may focus on solving only the real part of Eq. (67).

3.5 Structure of the ingoing-wave function to $O(\epsilon^2)$

With the boundary condition discussed in the previous section, we can integrate the ingoing-wave Regge-Wheeler function iteratively to higher orders of ϵ in the post-Minkowskian expansion, $\epsilon \ll 1$. This was carried out in [65] to $O(\epsilon^2)$ and in [77] to $O(\epsilon^3)$ (See [46] for details). Here, we do not recapitulate the details of the calculation since it is already quite involved at $O(\epsilon^2)$, with much less space for physical intuition. Instead, we describe the general properties of the ingoing-wave function to $O(\epsilon^2)$.

As discussed in the previous section, the ingoing wave Regge-Wheeler function X_ℓ can be made real up to $O(\epsilon^{2\ell+1})$, or to $O(\epsilon^5)$ of the post-Minkowski expansion if we recall $\ell \geq 2$. Choosing the phase of X_ℓ in this way, let us explicitly write down the expressions of $\text{Im}(\xi_\ell^{(n)})$ $(n = 1, 2)$ in terms of $\text{Re}(\xi_\ell^{(m)})$ $(m \leq n - 1)$. We decompose the real and imaginary parts of $\xi_\ell^{(n)}$ as

$$\xi_\ell^{(n)} = f_\ell^{(n)} + i g_\ell^{(n)}. \tag{78}$$

Inserting this expression into Eq. (54) and expanding the result with respect to ϵ, and noting $f_\ell^{(0)} = j_\ell$ and $g_\ell^{(0)} = 0$, we find

$$\begin{aligned}
X_\ell &= e^{-i\epsilon \ln(z-\epsilon)} z(j_\ell + \epsilon(f_\ell^{(1)} + ig_\ell^{(1)}) + \epsilon^2(f_\ell^{(2)} + ig_\ell^{(2)}) + \cdots) \\
&= z\left(j_\ell + \epsilon f_\ell^{(1)} + \epsilon^2\left(f_\ell^{(2)} + g_\ell^{(1)} \ln z - \frac{1}{2}j_\ell(\ln z)^2\right) + \cdots\right) \\
&\quad + iz\left(\epsilon(g_\ell^{(1)} - j_\ell \ln z) + \epsilon^2\left(g_\ell^{(2)} + \frac{1}{z}j_\ell - f_\ell^{(1)} \ln z\right) + \cdots\right). \tag{79}
\end{aligned}$$

Hence, we have

$$g_\ell^{(1)} = j_\ell \ln z \quad g_\ell^{(2)} = -\frac{1}{z} j_\ell + f_\ell^{(1)} \ln z, \cdots. \tag{80}$$

We thus have the post-Minkowski expansion of X_ℓ as

$$X_\ell = \sum_{n=0}^{\infty} \epsilon^n X_\ell^{(n)},$$

$$X_\ell^{(0)} = z\, j_\ell, \quad X_\ell^{(1)} = z f_\ell^{(1)}, \quad X_\ell^{(2)} = z \left(f_\ell^{(2)} + \frac{1}{2} j_\ell \, (\ln z)^2 \right), \cdots. \tag{81}$$

Now, let us consider the asymptotic behavior of X_ℓ at $z \ll 1$. As we know that $\xi_\ell^{(1)}$ and $\xi_\ell^{(2)}$ are regular at $z = 0$, it is readily obtained by simply assuming Taylor expansion forms for them (including possible $\ln z$ terms), inserting them to Eq. (67), and comparing the terms of the same order on both sides of the equation. We denote the right-hand side of Eq. (67) by $S_\ell^{(n)}$.

For $n = 1$, we have

$$\begin{aligned}
\mathrm{Re}\left(S_\ell^{(1)}\right) &= \frac{1}{z}\left(j_\ell'' + \frac{1}{z} j_\ell' - \frac{4 + z^2}{z^2} j_\ell \right) \\
&= \begin{cases} O(z) & \text{for } \ell = 2, \\ O(z^{\ell-3}) & \text{for } \ell \geq 3. \end{cases}
\end{aligned} \tag{82}$$

Inserting this to Eq. (67) with $n = 1$ we find

$$\mathrm{Re}\left(\xi_\ell^{(1)}\right) = f_\ell^{(1)} = \begin{cases} O(z^3) & \text{for } \ell = 2, \\ O(z^{\ell-1}) & \text{for } \ell \geq 3. \end{cases} \tag{83}$$

Of course, this behavior is consistent with the full post-Minkowski solution given in Eq. (70).

For $n = 2$, we then have

$$\begin{aligned}
\mathrm{Re}\left(S_\ell^{(2)}\right) &= \frac{1}{z}\left(f_\ell^{(1)\prime\prime} + \frac{1}{z} f_\ell^{(1)\prime} - \frac{4 + z^2}{z^2} f_\ell^{(1)} \right) - \frac{1}{z}\left(2 g_\ell^{(1)\prime} + \frac{1}{z} g_\ell^{(1)} \right) \\
&= -\frac{1}{z}(j_\ell \ln z)' - \frac{1}{z^2} j_\ell \ln z + \begin{cases} O(z^{\ell-2}) & \text{for } \ell = 2, 3, \\ O(z^{\ell-4}) & \text{for } \ell \geq 4, \end{cases}
\end{aligned} \tag{84}$$

This gives

$$\mathrm{Re}\left(\xi_\ell^{(2)}\right) = f_\ell^{(2)} = \begin{cases} O(z^\ell) + O(z^\ell) \ln z - \frac{1}{2} j_\ell (\ln z)^2 & \text{for } \ell = 2, 3, \\ O(z^{\ell-2}) + O(z^\ell) \ln z - \frac{1}{2} j_\ell (\ln z)^2 & \text{for } \ell \geq 4. \end{cases} \tag{85}$$

Note that the $\ln z$ terms in (84) arising from $g_\ell^{(1)}$ gives the $(\ln z)^2$ term in $f_\ell^{(2)}$ that just cancels the $j_\ell (\ln z)^2/2$ term of $X_\ell^{(2)}$ in Eq. (81).

Inserting Eqs. (83) and (85) into the relevant expressions in Eq. (81), we find

$$X_2 = z^3 \left[O(1) + \epsilon\, O(z) + \epsilon^2 \left\{ O(1) + O(1) \ln z \right\} + \cdots \right],$$

$$X_3 = z^3 \left[O(z) + \epsilon\, O(1) + \epsilon^2 \left\{ O(z) + O(z)\ln z \right\} + \cdots \right], \tag{86}$$

$$X_\ell = z^3 \left[O(z^{\ell-2}) + \epsilon\, O(z^{\ell-3}) + \epsilon^2 \left\{ O(z^{\ell-4}) + O(z^{\ell-2})\ln z \right\} + \cdots \right] \quad (\ell \geq 4).$$

Note that, for $\ell = 2$ and 3, the leading behavior of $X_\ell^{(n)}$ at $n = \ell - 1$ is more regular than the naively expected behavior, $\sim z^{\ell+1-n}$, which propagates to the consecutive higher order terms in ϵ. This behavior seems to hold for general ℓ, but we do not know a physical explanation of it.

Given a post-Newtonian order to which we want to calculate, by setting $z = O(v)$ and $\epsilon = O(v^3)$, the above asymptotic behaviors tell us the highest order of $X_\ell^{(n)}$ we need. We also see the presence of $\ln z$ terms in $X_\ell^{(2)}$. The logarithmic terms appear as a consequence of the mathematical structure of the Regge-Wheeler equation at $z \ll 1$. The simple power series expansion of $X_\ell^{(n)}$ in terms of z breaks down at $O(\epsilon^2)$, and we have to add logarithmic terms to obtain the solution. These logarithmic terms will give rise to $\ln v$ terms in the waveform and luminosity formulas at infinity, beginning at $O(v^6)$ [72, 73]. It is not easy to explain physically how these $\ln v$ terms appear. But the above analysis suggests that the $\ln v$ terms in the luminosity originate from some spatially local curvature effects in the near zone.

Now we turn to the asymptotic behavior at $z = \infty$. For this purpose, let the asymptotic form of $f_\ell^{(n)}$ be

$$f_\ell^{(n)} \to P_\ell^{(n)} j_\ell + Q_\ell^{(n)} n_\ell \quad \text{as } z \to \infty. \tag{87}$$

Noting Eq. (81) and the equality $e^{-i\epsilon \ln(z-\epsilon)} = e^{-iz^*} e^{iz}$, the asymptotic form of X_ℓ is expressed as

$$X_\ell \to A_\ell^{\mathrm{inc}} e^{-i(z^* - \epsilon \ln \epsilon)} + A_\ell^{\mathrm{ref}} e^{i(z^* - \epsilon \ln \epsilon)},$$

$$A_\ell^{\mathrm{inc}} = \frac{1}{2} i^{\ell+1} e^{-i\epsilon \ln \epsilon} \left[1 + \epsilon \left\{ P_\ell^{(1)} + i \left(Q_\ell^{(1)} + \ln z \right) \right\} \right.$$
$$\left. + \epsilon^2 \left\{ \left(P_\ell^{(2)} - Q_\ell^{(1)} \ln z \right) + i \left(Q_\ell^{(2)} + P_\ell^{(1)} \ln z \right) \right\} + \cdots \right]. \tag{88}$$

Note that

$$\omega r^* = \omega \left(r + 2M \ln \frac{r - 2M}{2M} \right) = z^* - \epsilon \ln \epsilon,$$

because of our definition of z^*, $z^* = z + \epsilon + \ln(z - \epsilon)$. The phase factor $e^{-i\epsilon \ln \epsilon}$ of A_ℓ^{inc} originates from this definition, but it represents a physical phase shift due to wave propagation on the curved background.

As one may immediately notice, the above expression for A_ℓ^{inc} contains $\ln z$-dependent terms. Since A_ℓ^{inc} should be constant, $P_\ell^{(n)}$ and $Q_\ell^{(n)}$ should contain appropriate $\ln z$-dependent terms which exactly cancel the $\ln z$-dependent terms in the formula (88). To be explicit, we must have

$$P_\ell^{(1)} = p_\ell^{(1)},$$
$$Q_\ell^{(1)} = q_\ell^{(1)} - \ln z,$$

$$
\begin{aligned}
P_\ell^{(2)} &= p_\ell^{(2)} + q_\ell^{(1)} \ln z - (\ln z)^2 \,, \\
Q_\ell^{(2)} &= q_\ell^{(2)} - p_\ell^{(1)} \ln z \,,
\end{aligned}
\tag{89}
$$

where $p_\ell^{(n)}$ and $q_\ell^{(n)}$ are constants. These relations can be used to check the consistency of the solution $f^{(n)}$ obtained by integration. In terms of $p_\ell^{(n)}$ and $q_\ell^{(n)}$, A_ℓ^{inc} is expressed as

$$
A_\ell^{\mathrm{inc}} = \frac{1}{2} i^{\ell+1} e^{-i\epsilon \ln \epsilon} \left[1 + \epsilon \left(p_\ell^{(1)} + i q_\ell^{(1)} \right) + \epsilon^2 \left(p_\ell^{(2)} + i q_\ell^{(2)} \right) + \cdots \right].
\tag{90}
$$

Note that the above form of A_ℓ^{inc} implies that the so-called tail of radiation, which is due to the curvature scattering of waves, will contain $\ln v$ terms as phase shifts in the waveform, but will not give rise to such terms in the luminosity formula. This supports our previous argument on the origin of the $\ln v$ terms in the luminosity. That is, it is not due to the wave propagation effect but due to some near-zone curvature effect.

4 Analytic solutions of the homogeneous Teukolsky equation by means of the series expansion of special functions

In this section, we review a method developed by Mano, Suzuki and Takasugi [43] who found analytic expressions of the solutions of the homogeneous Teukolsky equation. In this method, the exact solutions of the radial Teukolsky equation (9) are expressed in two kinds of series expansions. One is given by a series of hypergeometric functions and the other by a series of the Coulomb wave functions. The former is convergent at horizon and the latter at infinity. The matching of these two solutions are done exactly in the overlapping region of convergence. They also found that the series expansions are naturally related to the low frequency expansion. Properties of the analytic solutions were studied in detail in [45]. Thus, the formalism is quite powerful when dealing with the post-Newtonian expansion, especially at higher orders.

In many cases when we study the perturbation of a Kerr black hole, it is more convenient to use the Sasaki-Nakamura equation since it has the form of a standard wave equation, similar to the Regge-Wheeler equation. However, it is not quite suited for investigating analytic properties of the solution near the horizon. In contrast, the Mano-Suzuki-Takasugi (MST) formalism allows us to investigate analytic properties of the solution near the horizon systematically. Hence, it can be used to compute the higher order post-Newtonian terms of the gravitational waves absorbed into a rotating black hole.

We also note that this method is the only existing method that can be used to calculate the gravitational waves emitted to infinity to an arbitrarily high post-Newtonian order in principle.

4.1 Angular eigenvalue

The solutions of the angular equation (10) that reduce to the spin-weighted spherical harmonics in the limit $a\omega \to 0$ are called the spin-weighted spheroidal harmonics. They are the eigen functions of Eq. (10) with λ being the eigenvalues. The eigenvalues λ are necessary for discussions of the radial Teukolsky equation. For general spin weight s, the spin weighted spheroidal harmonics obeys

$$
\left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left\{ \sin\theta \frac{d}{d\theta} \right\} - a^2\omega^2 \sin^2\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} \right.
$$
$$
\left. - 2a\omega s\cos\theta + s + 2ma\omega + \lambda \right]_s S_{\ell m} = 0. \tag{91}
$$

In the post-Newtonian expansion, the parameter $a\omega$ is assumed to be small. Then, it is straightforward to obtain a spheroidal harmonic $_sS_{\ell m}$ of spin-weight s and its eigenvalue λ perturbatively by the standard method [60, 74, 69].

It is also possible to obtain the spheroidal harmonics by expansion in terms of the Jacobi functions [29]. In this method, if we calculate numerically, we can obtain them and their eigenvalues for arbitrary value of $a\omega$.

Here we only show an analytic formula for the eigenvalue λ accurate to $O\left((a\omega)^2\right)$, which is needed for the calculation of the radial functions. It is given by

$$
\lambda = \lambda_0 + a\omega\lambda_1 + a^2\omega^2\lambda_2 + O((a\omega)^3) \tag{92}
$$

where

$$
\begin{aligned}
\lambda_0 &= \ell(\ell+1) - s(s+1), \\
\lambda_1 &= -2m \left(1 + \frac{s^2}{\ell(\ell+1)} \right), \\
\lambda_2 &= 1 + (H(\ell+1) - H(\ell) - 1),
\end{aligned} \tag{93}
$$

with

$$
H(\ell) = \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell - 1)\ell^3(2\ell + 1)}. \tag{94}
$$

4.2 Horizon solution in series of hypergeometric functions

As in Sec. 3, we focus on the ingoing-wave function of the radial Teukolsky equation (9). Since the analysis below is applicable to any spin, $|s| = 0, 1/2, 1, 3/2$ and 2, we do not specify it except when it is needed. Also, the analysis is not restricted to the case $a\omega \ll 1$ unless so stated explicitly. For general spin weight s, the homogeneous Teukolsky equation is given by

$$
\Delta^{-s} \frac{d}{dr} \left(\Delta^{s+1} \frac{dR_{\ell m\omega}}{dr} \right) + \left(\frac{K^2 - 2is(r - M)K}{\Delta} + 4is\omega r - \lambda \right) R_{\ell m\omega} = 0. \tag{95}
$$

As before, taking account of the symmetry $\bar{R}_{\ell m\omega} = R_{\ell - m - \omega}$, we may assume $\epsilon = 2M\omega > 0$ if necessary.

The Teukolsky equation has two regular singularities at $r = r_\pm$, and one irregular singularity at $r = \infty$. This implies that it cannot be represented in the form of a single hypergeometric equation. However, if we focus on the solution near the horizon, it may be approximated by a hypergeometric equation. This motivates us to consider the solution expressed in terms of a series of hypergeometric functions.

We define the independent variable x in place of z ($= \omega r$) as

$$x = \frac{z_+ - z}{\epsilon \kappa},\tag{96}$$

where

$$z_\pm = \omega r_\pm, \quad \kappa = \sqrt{1 - q^2}, \quad q = \frac{a}{M}.$$

For later convenience, we also introduce $\tau = (\epsilon - mq)/\kappa$ and $\epsilon_\pm = (\epsilon \pm \tau)/2$. Taking into account the structure of the singularities at $r = r_\pm$, we put the ingoing-wave Teukolsky function $R^{\mathrm{in}}_{\ell m \omega}$ as

$$R^{\mathrm{in}}_{\ell m \omega} = e^{i\epsilon \kappa x}(-x)^{-s - i(\epsilon + \tau)/2}(1 - x)^{i(\epsilon - \tau)/2} p_{\mathrm{in}}(x).\tag{97}$$

Then the radial Teukolsky equation becomes

$$\begin{aligned}
x(1 - x)p_{\mathrm{in}}'' &+ [1 - s - i\epsilon - i\tau - (2 - 2i\tau)x]p_{\mathrm{in}}' \\
&+ [i\tau(1 - i\tau) + \lambda + s(s + 1)]p_{\mathrm{in}} \\
&= 2i\epsilon\kappa[-x(1 - x)p_{\mathrm{in}}' + (1 - s + i\epsilon - i\tau)x\, p_{\mathrm{in}}] \\
&\quad + [\epsilon^2 - i\epsilon\kappa(1 - 2s)]p_{\mathrm{in}},
\end{aligned}\tag{98}$$

where a prime denotes d/dx. The left-hand-side of Eq. (98) is in the form of a hypergeometric equation. In the limit $\epsilon \to 0$, noting Eq. (92), we find that a solution which is finite at $x = 0$ is given by

$$p_{\mathrm{in}}(\epsilon \to 0) = F(-\ell - i\tau, \ell + 1 - i\tau, 1 - s - i\tau, x).\tag{99}$$

For a general value of ϵ, Eq. (98) suggests that a solution may be expanded in a series of hypergeometric functions with ϵ being a kind of an expansion parameter. This idea was extensively developed by Leaver [42]. Leaver obtained solutions of the Teukolsky equation expressed in a series of the Coulomb wave functions. The MST formalism is an elegant reformulation of the one by Leaver [42].

The essential point is to introduce the so-called renormalized angular momentum ν which is a generalization of ℓ to a non-integer value such that the Teukolsky equation admits a solution in a convergent series of hypergeometric functions. Namely, we add the term $[\nu(\nu + 1) - \lambda - s(s + 1)]p_{\mathrm{in}}$ to both sides of Eq. (98) to rewrite it as

$$\begin{aligned}
x(1 - x)p_{\mathrm{in}}'' &+ [1 - s - i\epsilon - i\tau - (2 - 2i\tau)x]p_{\mathrm{in}}' \\
&+ [i\tau(1 - i\tau) + \nu(\nu + 1)]p_{\mathrm{in}} \\
&= 2i\epsilon\kappa[-x(1 - x)p_{\mathrm{in}}' + (1 - s + i\epsilon - i\tau)x\, p_{\mathrm{in}}] \\
&\quad + [\nu(\nu + 1) - \lambda - s(s + 1) + \epsilon^2 - i\epsilon\kappa(1 - 2s)]p_{\mathrm{in}}.
\end{aligned}\tag{100}$$

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Of course, no modification is done to the original equation, and ν is just an irrelevant parameter at this stage. A trick is to consider the right-hand-side of the above equation as a perturbation, and look for a formal solution specified by the index ν in a series expansion form. Then, only after we obtain the formal solution, we require that the series should converge and this requirement determines the value of ν. Note that, if we take the limit $\epsilon \to 0$, we must have $\nu \to \ell$ (or $\nu \to -\ell - 1$), to assure $[\nu(\nu + 1) - \lambda - s(s + 1)] \to 0$ and to recover the solution (99)

Let us denote the formal solution specified by a value of ν by $p_{\rm in}^{\nu}$. We express it in the series form,

$$p_{\rm in}^{\nu} = \sum_{n=-\infty}^{\infty} a_n \, p_{n+\nu}(x), \tag{101}$$

$$p_{n+\nu}(x) = F(n + \nu + 1 - i\tau, -n - \nu - i\tau; 1 - s - i\epsilon - i\tau; x). \tag{102}$$

Here, the hypergeometric functions $p_{n+\nu}(x)$ satisfy the recurrence relations [43],

$$
\begin{aligned}
xp_{n+\nu} &= -\frac{(n + \nu + 1 - s - i\epsilon)(n + \nu + 1 - i\tau)}{2(n + \nu + 1)(2n + 2\nu + 1)} p_{n+\nu+1} \\
&+ \frac{1}{2}\left[1 + \frac{i\tau(s + i\epsilon)}{(n + \nu)(n + \nu + 1)}\right] p_{n+\nu} \\
&- \frac{(n + \nu + s + i\epsilon)(n + \nu + i\tau)}{2(n + \nu)(2n + 2\nu + 1)} p_{n+\nu-1},
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
x(1 - x)p'_{n+\nu} &= \frac{(n + \nu + i\tau)(n + \nu + 1 - i\tau)(n + \nu + 1 - s - i\epsilon)}{2(n + \nu + 1)(2n + 2\nu + 1)} p_{n+\nu+1} \\
&+ \frac{1}{2}(s + i\epsilon)\left[1 + \frac{i\tau(1 - i\tau)}{(n + \nu)(n + \nu + 1)}\right] p_{n+\nu} \\
&- \frac{(n + \nu + 1 - i\tau)(n + \nu + i\tau)(n + \nu + s + i\epsilon)}{2(n + \nu)(2n + 2\nu + 1)} p_{n+\nu-1},
\end{aligned} \tag{104}
$$

Inserting the series (101) into Eq. (100) and using the above recurrence relations, we obtain a three-term recurrence relation among the expansion coefficients a_n. It is given by

$$\alpha_n^{\nu} a_{n+1} + \beta_n^{\nu} a_n + \gamma_n^{\nu} a_{n-1} = 0, \tag{105}$$

where

$$
\begin{aligned}
\alpha_n^{\nu} &= \frac{i\epsilon\kappa(n + \nu + 1 + s + i\epsilon)(n + \nu + 1 + s - i\epsilon)(n + \nu + 1 + i\tau)}{(n + \nu + 1)(2n + 2\nu + 3)}, \\
\beta_n^{\nu} &= -\lambda - s(s + 1) + (n + \nu)(n + \nu + 1) + \epsilon^2 + \epsilon(\epsilon - mq) \\
&\quad + \frac{\epsilon(\epsilon - mq)(s^2 + \epsilon^2)}{(n + \nu)(n + \nu + 1)}, \\
\gamma_n^{\nu} &= -\frac{i\epsilon\kappa(n + \nu - s + i\epsilon)(n + \nu - s - i\epsilon)(n + \nu - i\tau)}{(n + \nu)(2n + 2\nu - 1)}.
\end{aligned} \tag{106}
$$

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The convergence of the series (101) is determined by the asymptotic behaviors of the coefficients a_n^ν at $n \to \pm\infty$. We thus discuss properties of the three-term recurrence relation (105), and the role of the parameter ν in detail.

The general solution of the recurrence relation (105) is expressed in terms of two linearly independent solutions $\{f_n^{(1)}\}$ and $\{f_n^{(2)}\}$ ($n = \pm 1, \pm 2, \cdots$). According to the theory of three-term recurrence relations ([35], p.31) when there exists a pair of solutions which satisfy

$$\lim_{n\to\infty} \frac{f_n^{(1)}}{f_n^{(2)}} = 0 \quad \left(\lim_{n\to-\infty} \frac{f_n^{(1)}}{f_n^{(2)}} = 0 \right), \tag{107}$$

then the solution $\{f_n^{(1)}\}$ is called *minimal* as $n \to \infty$ ($n \to -\infty$). Any non-minimal solution is called *dominant*. The minimal solution (either as $n \to \infty$ or as $n \to -\infty$) is determined uniquely up to an overall normalization factor.

The three-term recurrence relation is closely related to continued fractions. We introduce

$$R_n \equiv \frac{a_n}{a_{n-1}}, \quad L_n \equiv \frac{a_n}{a_{n+1}}. \tag{108}$$

We can express R_n and L_n in terms of continued fractions as

$$\begin{aligned}
R_n &= -\frac{\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}} \\
&= -\frac{\gamma_n^\nu}{\beta_n^\nu -} \frac{\alpha_n \gamma_{n+1}^\nu}{\beta_{n+1}^\nu -} \frac{\alpha_{n+1} \gamma_{n+2}^\nu}{\beta_{n+2}^\nu -} \cdots, \tag{109} \\
L_n &= -\frac{\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}} \\
&= -\frac{\alpha_n^\nu}{\beta_n^\nu -} \frac{\alpha_{n-1} \gamma_n^\nu}{\beta_{n-1}^\nu -} \frac{\alpha_{n-2} \gamma_{n-1}^\nu}{\beta_{n-2}^\nu -} \cdots. \tag{110}
\end{aligned}$$

These expressions for R_n and L_n are valid if the respective continued fractions converge. It is proved ([35], p.31) that the continued fraction (109) converges if and only if the recurrence relation (105) possesses a minimal solution as $n \to \infty$, and the same for the continued fraction (110) as $n \to -\infty$.

Analysis of the asymptotic behavior of (105) shows that, as long as ν is finite, there exists a set of two independent solutions which behave as (e.g., [35], p.35),

$$\lim_{n\to\infty} n\frac{a_n^{(1)}}{a_{n-1}^{(1)}} = \frac{i\epsilon\kappa}{2}, \quad \lim_{n\to\infty} \frac{a_n^{(2)}}{na_{n-1}^{(2)}} = \frac{2i}{\epsilon\kappa}, \tag{111}$$

and another set of two independent solutions which behave as

$$\lim_{n\to-\infty} n\frac{b_n^{(1)}}{b_{n+1}^{(1)}} = -\frac{i\epsilon\kappa}{2}, \quad \lim_{n\to-\infty} \frac{b_n^{(2)}}{nb_{n+1}^{(2)}} = -\frac{2i}{\epsilon\kappa}. \tag{112}$$

Thus $\{a_n^{(1)}\}$ is minimal as $n \to \infty$ and $\{b_n^{(1)}\}$ is minimal as $n \to -\infty$.

Since the recurrence relation (105) possesses minimal solutions as $n \to \pm\infty$, the continued fractions on the right-hand-sides of Eqs. (109) and (110) converge for $a_n = a_n^{(1)}$ and $a_n = b_n^{(1)}$. In general, however, $a_n^{(1)}$ and $b_n^{(1)}$ do not coincide. Here, we use the freedom of ν to obtain a consistent solution. Let $\{f_n^\nu\}$ be a sequence which is minimal for both $n \to \pm\infty$. We then have expressions for f_n^ν/f_{n-1}^ν and f_n^ν/f_{n+1}^ν in terms of continued fractions as

$$\tilde{R}_n \equiv \frac{f_n}{f_{n-1}} = -\frac{\gamma_n^\nu}{\beta_n^\nu-} \frac{\alpha_n \gamma_{n+1}^\nu}{\beta_{n+1}^\nu-} \frac{\alpha_{n+1} \gamma_{n+2}^\nu}{\beta_{n+2}^\nu-} \cdots, \tag{113}$$

$$\tilde{L}_n \equiv \frac{f_n}{f_{n+1}} = -\frac{\alpha_n^\nu}{\beta_n^\nu-} \frac{\alpha_{n-1} \gamma_n^\nu}{\beta_{n-1}^\nu-} \frac{\alpha_{n-2} \gamma_{n-1}^\nu}{\beta_{n-2}^\nu-} \cdots. \tag{114}$$

This implies

$$\tilde{R}_n \tilde{L}_{n-1} = 1 . \tag{115}$$

Thus, if we choose ν such that it satisfies the implicit equation for ν, Eq. (115), for a certain n, we obtain a unique minimal solution $\{f_n^\nu\}$ which is valid over the entire range of n, $-\infty < n < \infty$, that is

$$\lim_{n\to\infty} n\frac{f_n^\nu}{f_{n-1}^\nu} = \frac{i\epsilon\kappa}{2}, \quad \lim_{n\to-\infty} n\frac{f_n^\nu}{f_{n+1}^\nu} = -\frac{i\epsilon\kappa}{2} . \tag{116}$$

Note that if Eq. (115) for a certain value of n is satisfied, it is automatically satisfied for any other value of n.

The minimal solution is also important for the convergence of the series (101). For the minimal solution $\{f_n^\nu\}$, together with the properties of the hypergeometric functions $p_{n+\nu}$ for large $|n|$, we find

$$\lim_{n\to\infty} n\frac{f_{n+1}^\nu p_{n+\nu+1}(x)}{f_n^\nu p_{n+\nu}(x)} = -\lim_{n\to-\infty} n\frac{f_{n-1}^\nu p_{n+\nu-1}(x)}{f_n^\nu p_{n+\nu}(x)}$$
$$= \frac{i\epsilon\kappa}{2}[1 - 2x + ((1-2x)^2 - 1)^{1/2}]. \tag{117}$$

Thus the series of hypergeometric functions (101) converges for all x in the range $0 \geq x > -\infty$ (in fact, for all complex values of x except at $|x| = \infty$), provided the coefficients are given by the minimal solution.

Instead of Eq. (115), we may consider an equivalent but practically more convenient form of an equation that determines the value of ν. Dividing Eq. (105) by a_n, we find

$$\beta_n^\nu + \alpha_n^\nu R_{n+1} + \gamma_n^\nu L_{n-1} = 0, \tag{118}$$

where R_{n+1} and L_{n+1} are those given by the continued fractions (113) and (114), respectively. Although the value of n in this equation is arbitrary, it is convenient to set $n = 0$ to solve for ν.

For later use, we need a series expression for R^{in} with better convergence properties at large $|x|$. Using analytic properties of hypergeometric functions, we have

$$R^{\mathrm{in}} = R_0^\nu + R_0^{-\nu-1}, \tag{119}$$

where

$$
\begin{aligned}
R_0^\nu &= e^{i\epsilon\kappa x}(-x)^{-s-\frac{i}{2}(\epsilon+\tau)}(1-x)^{\frac{i}{2}(\epsilon+\tau)+\nu} \\
&\times \sum_{n=-\infty}^{\infty} f_n^\nu \frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\epsilon)} \\
&\times (1-x)^n F(-n-\nu-i\tau, -n-\nu-s-i\epsilon; -2n-2\nu; \frac{1}{1-x}). \quad (120)
\end{aligned}
$$

This expression explicitly exhibits the symmetry of R_{in} under the interchange of ν and $-\nu-1$. This is a result of the fact that $\nu(\nu+1)$ is invariant under the interchange $\nu \leftrightarrow -\nu-1$. Accordingly, the recurrence relation (105) has the structure that $f_{-n}^{-\nu-1}$ satisfies the same recurrence relation as f_n^ν.

Finally, we note that if ν is a solution of Eq. (115) or (118), $\nu+k$ with an arbitrary integer k is also a solution, since ν appears only in the combination of $n+\nu$. Thus, Eq. (115) or (118) contains infinite number of roots. However, not all of these can be used to express a solution we want. As noted in the earlier part of this subsection, in order to reproduce the solution in the limit $\epsilon \to 0$, Eq. (99), we must have $\nu \to \ell$ (or $\nu \to -\ell-1$ by symmetry). We thus impose a constraint on ν such that it must continuously approach ℓ as $\epsilon \to 0$.

4.3   Outer solution in series of Coulomb wave functions

The solution in series of hypergeometric functions discussed in the previous subsection is convergent at any finite value of r. However, it does not converge at infinity, and hence the asymptotic amplitudes, B^{inc} and B^{ref}, cannot be determined from it. To determine the asymptotic amplitudes, it is necessary to construct a solution which is valid at infinity and match the two solutions in a region where both solutions converge. The solution convergent at infinity was obtained by Leaver in series of Coulomb wave functions [42]. In this subsection, we review Leaver's solution based on [45].

In this subsection again, by noting the symmetry $\bar{R}_{\ell m \omega} = R_{\ell-m-\omega}$, we assume $\omega > 0$ without loss of generality.

First, we define a variable $\hat{z} = \omega(r-r_-) = \epsilon\kappa(1-x)$. Let us denote a Teukolsky function by R_C. We introduce a function $f(\hat{z})$ by

$$
R_C = \hat{z}^{-1-s}\left(1-\frac{\epsilon\kappa}{\hat{z}}\right)^{-s-i(\epsilon+\tau)/2} f(\hat{z}). \quad (121)
$$

Then the Teukolsky equation becomes

$$
\begin{aligned}
\hat{z}^2 f'' &+ [\hat{z}^2 + (2\epsilon+2is)\hat{z} - \lambda - s(s+1)]f \\
&= \epsilon\kappa\hat{z}(f''+f) + \epsilon\kappa(s-1+2i\epsilon)f' \\
&\quad -\frac{\epsilon}{\hat{z}}(\kappa-i(\epsilon-mq))(s-1+i\epsilon)f \\
&\quad +[-2\epsilon^2+\epsilon mq+\kappa(\epsilon^2+i\epsilon s)]f. \quad (122)
\end{aligned}
$$

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We see that the right-hand-side is explicitly of $O(\epsilon)$ and the left-hand-side is in the form of the Coulomb wave equation. Therefore, in the limit $\epsilon \to 0$, we obtain a solution,

$$f(\hat{z}) = F_\ell(-is - \epsilon, \hat{z}), \qquad (123)$$

where $F_L(\eta, \hat{z})$ is a Coulomb wave function given by

$$F_L(\eta, \hat{z}) = e^{-i\hat{z}} 2^L \hat{z}^{L+1} \frac{\Gamma(L+1-i\eta)}{\Gamma(2L+2)} \Phi(L+1-i\eta, 2L+2; 2i\hat{z}), \qquad (124)$$

and Φ is the regular confluent hypergeometric function function (see [7], section 13) which is regular at $\hat{z} = 0$.

In the same spirit as in the previous subsection, we introduce the renormalized angular momentum ν. That is, we add $[\lambda + s(s+1) - \nu(\nu+1)]f(\hat{z})$ to both sides of Eq. (122) to rewrite it as

$$\hat{z}^2 f'' + [\hat{z}^2 + (2\epsilon + 2is)\hat{z} - \nu(\nu+1)]f$$
$$= \epsilon\kappa\hat{z}(f'' + f) + \epsilon\kappa(s - 1 + 2i\epsilon_+)f' - \frac{\epsilon}{\hat{z}}(\kappa - i(\epsilon - mq))(s - 1 + i\epsilon)f$$
$$+ [-\nu(\nu+1) + \lambda + s(s+1) - 2\epsilon^2 + \epsilon mq + \kappa(\epsilon^2 + i\epsilon s)]f. \qquad (125)$$

We denote the formal solution specified by the index ν by $f_\nu(\hat{z})$, and expand it in terms of the Coulomb wave functions as

$$f_\nu = \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} b_n F_{n+\nu}(-is - \epsilon, \hat{z}), \qquad (126)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$. Then, using the recurrence relations among $F_{n+\nu}$,

$$\frac{1}{z} F_{n+\nu} = \frac{(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)} F_{n+\nu+1} + \frac{is+\epsilon}{(n+\nu)(n+\nu+1)} F_{n+\nu}$$
$$+ \frac{(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)} F_{n+\nu-1}, \qquad (127)$$

$$F'_{n+\nu} = -\frac{(n+\nu)(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)} F_{n+\nu+1} + \frac{is+\epsilon}{(n+\nu)(n+\nu+1)} F_{n+\nu}$$
$$+ \frac{(n+\nu+1)(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)} F_{n+\nu-1}, \qquad (128)$$

we can derive the recurrence relation among b_n. The result turns out to be identical to the one given by Eq. (105) for a_n. We mention that the extra factor $(\nu+1+s-i\epsilon)_n/(\nu+1-s+i\epsilon)_n$ in Eq. (126) is introduced to make the recurrence relation exactly identical to Eq. (105).

The fact that we have the same recurrence relation as Eq. (105) implies that if we choose the parameter ν in Eq. (126) to be the same as the one given by a solution of Eq. (115) or (118), the sequence $\{f_n^\nu\}$ is also the solution for $\{b_n\}$ which is minimal for both $n \to \pm\infty$. Let us set

$$g_n^\nu = (-i)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} f_n^\nu.$$

By choosing ν as stated above, we have the asymptotic value for the ratio of two successive terms of g_n^ν as

$$\lim_{n\to\infty} n\, \frac{g_n^\nu}{g_{n-1}^\nu} = \lim_{n\to-\infty} n\, \frac{g_n^\nu}{g_{n+1}^\nu} = \frac{\epsilon\kappa}{2}\,. \tag{129}$$

Using an asymptotic property of the Coulomb wave functions, we have

$$\lim_{n\to\infty} \frac{g_n^\nu F_{n+\nu}(z)}{g_{n-1}^\nu F_{n+\nu-1}(z)} = \lim_{n\to-\infty} \frac{g_n^\nu F_{n+\nu}(z)}{g_{n-1}^\nu F_{n+\nu+1}(z)} = \frac{\epsilon\kappa}{z}\,. \tag{130}$$

We thus find that the series (126) converges at $\hat{z} > \epsilon\kappa$ or equivalently $r > r_+$.

The fact that we can use the same ν as in the case of hypergeometric functions to obtain the convergence of the series of the Coulomb wave functions is crucial to match the horizon and outer solutions.

Here, we note an analytic property of the confluent hypergeometric function ([28],p.259),

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{ia\pi} \Psi(a,c;x) + \frac{\Gamma(c)}{\Gamma(a)} e^{i\pi(a-c)} e^x \Psi(c-a,c;-x), \tag{131}$$

where Ψ is the irregular confluent hypergeometric function, and $\mathrm{Im}(x) > 0$ is assumed. Using this with the identifications,

$$
\begin{aligned}
a &= n + \nu + 1 - s + i\epsilon\,, \\
c &= 2(n + \nu + 1)\,, \\
x &= 2i\hat{z}\,,
\end{aligned}
$$

we can rewrite R_C^ν (for $\omega > 0$) as

$$R_C^\nu = R_+^\nu + R_-^\nu, \tag{132}$$

where

$$
\begin{aligned}
R_+^\nu &= 2^\nu e^{-\pi\epsilon} e^{i\pi(\nu+1-s)} \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s-i\epsilon)} e^{-i\hat{z}} \hat{z}^{\nu+i\epsilon_+} (\hat{z}-\epsilon\kappa)^{-s-i\epsilon_+} \\
&\quad \times \sum_{n=-\infty}^{\infty} i^n f_n^\nu (2\hat{z})^n \Psi(n+\nu+1-s+i\epsilon, 2n+2\nu+2; 2i\hat{z}),
\end{aligned}
$$

$$\tag{133}$$

$$
\begin{aligned}
R_-^\nu &= 2^\nu e^{-\pi\epsilon} e^{-i\pi(\nu+1+s)} e^{i\hat{z}} \hat{z}^{\nu+i\epsilon_+} (\hat{z}-\epsilon\kappa)^{-s-i\epsilon_+} \\
&\quad \times \sum_{n=-\infty}^{\infty} i^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} f_n^\nu (2\hat{z})^n \\
&\quad \times \Psi(n+\nu+1+s-i\epsilon, 2n+2\nu+2; -2i\hat{z}). \tag{134}
\end{aligned}
$$

By noting an asymptotic behavior of $\Psi(a,c;x)$ at large $|x|$,

$$\Psi(a,c;x) \to x^{-a} \qquad \text{as } |x| \to \infty, \tag{135}$$

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we find

$$R_+^\nu = A_+^\nu z^{-1} e^{-i(z+\epsilon \ln z)}, \tag{136}$$

$$R_-^\nu = A_-^\nu z^{-1-2s} e^{i(z+\epsilon \ln z)}, \tag{137}$$

where

$$A_+^\nu = e^{-\frac{\pi}{2}\epsilon} e^{\frac{\pi}{2}i(\nu+1-s)} 2^{-1+s-i\epsilon} \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s-i\epsilon)} \sum_{n=-\infty}^{+\infty} f_n^\nu, \tag{138}$$

$$A_-^\nu = 2^{-1-s+i\epsilon} e^{-\frac{\pi}{2}i(\nu+1+s)} e^{-\frac{\pi}{2}\epsilon} \sum_{n=-\infty}^{+\infty} (-1)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} f_n^\nu. \tag{139}$$

We can see that the functions R_+^ν and R_-^ν are incoming-wave and outgoing-wave solutions at infinity, respectively. In particular, we have the upgoing solution, defined for $s = -2$ by the asymptotic behavior (15), expressed in terms of a series of Coulomb wave functions as

$$R^{\mathrm{up}} = R_-^\nu. \tag{140}$$

4.4 Matching of horizon and outer solutions

Now, we match the two types of solutions R_0^ν and R_C^ν. Note that both of them are convergent in a very large region of r, namely for $\epsilon \kappa < \hat{z} < \infty$. We see that both solutions behave a $\hat{z}^\nu$ multiplied by a single-valued function of $\hat{z}$ for large $|\hat{z}|$. Thus, the analytic properties of R_0^ν and R_C^ν are the same, which implies that these two are identical up to a constant multiple. Therefore, we set

$$R_0^\nu = K_\nu R_C^\nu. \tag{141}$$

In the region $\epsilon \kappa < \hat{z} < \infty$, we may expand both solutions in powers of $\hat{z}$ except for analytically non-trivial factors. We have

$$
\begin{aligned}
R_0^\nu &= e^{i\epsilon\kappa} e^{-i\hat{z}} (\epsilon\kappa)^{-\nu-i\epsilon_+} \hat{z}^{\nu+i\epsilon_+} \left(\frac{\hat{z}}{\epsilon\kappa} - 1\right)^{-s-i\epsilon_+} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} C_{n,j} \hat{z}^{n-j}, \\
&= e^{i\epsilon\kappa} e^{-i\hat{z}} (\epsilon\kappa)^{-\nu-i\epsilon_+} \hat{z}^{\nu+i\epsilon_+} \left(\frac{\hat{z}}{\epsilon\kappa} - 1\right)^{-s-i\epsilon_+} \sum_{k=-\infty}^{\infty} \sum_{n=k}^{\infty} C_{n,n-k} \hat{z}^k,
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
R_C^\nu &= e^{-i\hat{z}} 2^\nu (\epsilon\kappa)^{-s-i\epsilon_+} \hat{z}^{\nu+i\epsilon_+} \left(\frac{\hat{z}}{\epsilon\kappa} - 1\right)^{-s-i\epsilon_+} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} D_{n,j} \hat{z}^{n+j}, \\
&= e^{-i\hat{z}} 2^\nu (\epsilon\kappa)^{-s-i\epsilon_+} \hat{z}^{\nu+i\epsilon_+} \left(\frac{\hat{z}}{\epsilon\kappa} - 1\right)^{-s-i\epsilon_+} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{k} D_{n,k-n} \hat{z}^k,
\end{aligned}
\tag{143}
$$

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where

$$
\begin{aligned}
C_{n,j} &= \frac{\Gamma(1 - s - 2i\epsilon_+)\Gamma(2n + 2\nu + 1)}{\Gamma(n + \nu + 1 - i\tau)\Gamma(n + \nu + 1 - s - i\epsilon)} \\
&\quad \times \frac{(-n - \nu - i\tau)_j(-n - \nu - s - i\epsilon)_j}{(-2n - 2\nu)_j(j!)}(\epsilon\kappa)^{-n+j}f_n, \qquad (144)
\end{aligned}
$$

$$
\begin{aligned}
D_{n,j} &= (-1)^n(2i)^{n+j}\frac{\Gamma(n + \nu + 1 - s + i\epsilon)}{\Gamma(2n + 2\nu + 2)}\frac{(\nu + 1 + s - i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n} \\
&\quad \times \frac{(n + \nu + 1 - s + i\epsilon)_j}{(2n + 2\nu + 2)_j(j!)}f_n. \qquad (145)
\end{aligned}
$$

Then, by comparing each integer power of $\hat{z}$ in the summation, in the region $\epsilon\kappa \ll \hat{z} < \infty$, and using the formula $\Gamma(z)\Gamma(1 - z) = \pi/\sin\pi z$, we find

$$
\begin{aligned}
K_\nu &= e^{i\epsilon\kappa}(\epsilon\kappa)^{s-\nu}2^{-\nu}\left(\sum_{n=-\infty}^{r}D_{n,r-n}\right)^{-1}\left(\sum_{n=r}^{\infty}C_{n,n-r}\right) \\
&= \frac{e^{i\epsilon\kappa}(2\epsilon\kappa)^{s-\nu-r}2^{-s}i^r\Gamma(1 - s - 2i\epsilon_+)\Gamma(r + 2\nu + 2)}{\Gamma(r + \nu + 1 - s + i\epsilon)\Gamma(r + \nu + 1 + i\tau)\Gamma(r + \nu + 1 + s + i\epsilon)} \\
&\quad \times \left(\sum_{n=r}^{\infty}(-1)^n\frac{\Gamma(n + r + 2\nu + 1)}{(n-r)!}\frac{\Gamma(n + \nu + 1 + s + i\epsilon)}{\Gamma(n + \nu + 1 - s - i\epsilon)}\frac{\Gamma(n + \nu + 1 + i\tau)}{\Gamma(n + \nu + 1 - i\tau)}f_n^\nu\right) \\
&\quad \times \left(\sum_{n=-\infty}^{r}\frac{(-1)^n}{(r-n)!(r + 2\nu + 2)_n}\frac{(\nu + 1 + s - i\epsilon)_n}{(\nu + 1 - s + i\epsilon)_n}f_n^\nu\right)^{-1}, \qquad (146)
\end{aligned}
$$

where r can be any integer, and the factor K_ν should be independent of the choice of r. Although this fact is not manifest from Eq. (146), we can check it numerically, or analytically by expanding it in terms of ϵ.

We thus have two expressions for the ingoing-wave function R^{in}. One is given by Eq. (97) with p_{in}^ν expressed in terms of a series of hypergeometric functions as given by Eq. (101), which converges everywhere except at $r = \infty$. The other is expressed in terms of a series of Coulomb wave functions given by

$$
R^{\mathrm{in}} = K_\nu R_C^\nu + K_{-\nu-1}R_C^{-\nu-1}, \qquad (147)
$$

which converges at $r > r_+$, including $r = \infty$. Combining these two, we have a complete analytic solution for the ingoing-wave function.

Now we can obtain analytic expressions for the asymptotic amplitudes of R^{in}, B^{trans}, B^{inc} and B^{ref}. By investigating the asymptotic behaviors of the solution at $r \to \infty$ and $r \to r_+$, they are found to be

$$
B^{\mathrm{trans}} = \left(\frac{\epsilon\kappa}{\omega}\right)^{2s}e^{i\epsilon_+\ln\kappa}\sum_{n=-\infty}^{\infty}f_n^\nu. \qquad (148)
$$

$$
B^{\mathrm{inc}} = \omega^{-1}\left[K_\nu - ie^{-i\pi\nu}\frac{\sin\pi(\nu - s + i\epsilon)}{\sin\pi(\nu + s - i\epsilon)}K_{-\nu-1}\right]A_+^\nu e^{-i\epsilon\ln\epsilon}, \qquad (149)
$$

$$
B^{\mathrm{ref}} = \omega^{-1-2s}\left[K_\nu + ie^{i\pi\nu}K_{-\nu-1}\right]A_-^\nu e^{i\epsilon\ln\epsilon}. \qquad (150)
$$

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Incidentally, since we have the upgoing solution in the outer region (140), it is straightforward to obtain the asymptotic outgoing amplitude at infinity C^{trans} from Eq. (134). We find

$$C^{\text{trans}} = \omega^{-1-2s} e^{i\epsilon \ln \epsilon} A_-^{\nu}. \tag{151}$$

4.5 Low frequency expansion of hypergeometric expansion

So far, we have considered exact solutions of the Teukolsky equation. Now let us consider their low frequency approximation and determine the value of ν. We solve Eq. (118) with $n = 0$,

$$\beta_0^{\nu} + \alpha_0^{\nu} R_1 + \gamma_0^{\nu} L_{-1} = 0, \tag{152}$$

with a requirement that $\nu \to \ell$ as $\epsilon \to 0$.

To solve Eq. (152), we first note the following. Unless the value of ν is such that the denominator in the expression of α_n^{ν} or γ_n^{ν} happens to vanish or β_n^{ν} happens to vanish in the limit $\epsilon \to 0$, we have $\alpha_n^{\nu} = O(\epsilon)$, $\gamma_n^{\nu} = O(\epsilon)$ and $\beta_n^{\nu} = O(1)$. Also, from the asymptotic behavior of the minimal solution f_n^{ν} as $n \to \pm\infty$ given by Eq. (116), we have $R_n(\nu) = O(\epsilon)$ and $L_{-n}(\nu) = O(\epsilon)$ for sufficiently large n. Thus, except for exceptional cases mentioned above, the order of a_n^{ν} in ϵ increases as $|n|$ increases. That is, the series solution naturally gives the post-Minkowski expansion.

First let us consider the case of $R_n(\nu)$ for $n > 0$. It is easily seen that $\alpha_n^{\nu} = O(\epsilon)$, $\gamma_n^{\nu} = O(\epsilon)$ and $\beta_n^{\nu} = O(1)$ for all $n > 0$. Therefore, we have $R_n(\nu) = O(\epsilon)$ for all $n > 0$.

On the other hand, for $n < 0$, the order of $L_{-n}(\nu)$ behaves irregularly for certain values of n. For the moment, let us assume that $L_{-1}(\nu) = O(\epsilon)$. We see from Eqs. (106) that $\alpha_0^{\nu} = O(\epsilon)$, $\gamma_0^{\nu} = O(\epsilon)$, since $\nu = \ell + O(\epsilon)$. Then, Eq. (152) implies $\beta_0^{\nu} = O(\epsilon^2)$. Using the expansion of λ given by Eq. (92), we then find $\nu = \ell + O(\epsilon^2)$, i.e., there is no term of $O(\epsilon)$ in ν. With this estimate of ν, we see from Eq. (110) that $L_{-1}(\nu) = O(\epsilon)$ is justified if $L_{-2}(\nu)$ is of order unity or smaller.

The general behavior of the order of $L_{-n}(\nu)$ in ϵ for general values of s is rather complicated. However, if we assume s to be a non-integer and $\ell \geq |s|$, and $\tau = (\epsilon - mq)/\kappa = O(1)$, it is relatively easily studied. With the assumption that $\nu = \ell + O(\epsilon^2)$, we find there are three exceptional cases: For $n = -2\ell - 1$, we have $\alpha_n = O(\epsilon)$, $\beta = O(\epsilon^2)$ and $\gamma_n = O(\epsilon)$. For $n = -\ell - 1$, we have $\alpha_n^{\nu} = O(1/\epsilon)$ and $\beta_n^{\nu} = O(1/\epsilon)$ and $\gamma_n = O(\epsilon)$. And for $n = -\ell$, we have $\alpha_n = O(\epsilon)$, $\beta_n = O(1/\epsilon)$ and $\gamma_n = O(1/\epsilon)$. These implies consecutively that $L_{-2\ell-1}(\nu) = O(1/\epsilon)$, $L_{-\ell-1}(\nu) = O(1)$ and $L_{-\ell}(\nu) = O(\epsilon^2)$. To summarize, we have

$$R_n(\nu) = \frac{f_n^{\nu}}{f_{n-1}^{\nu}} = O(\epsilon) \quad \text{for all } n > 0,$$

$$L_{-\ell}(\nu) = \frac{f^{\nu}_{-\ell}}{f^{\nu}_{-\ell+1}} = O(\epsilon^2), \quad L_{-\ell-1}(\nu) = \frac{f^{\nu}_{-\ell-1}}{f^{\nu}_{-\ell}} = O(1),$$

$$L_{-2\ell-1}(\nu) = \frac{f^{\nu}_{-2\ell-1}}{f^{\nu}_{-2\ell}} = O(1/\epsilon),$$

$$L_n(\nu) = \frac{f^{\nu}_n}{f^{\nu}_{n+1}} = O(\epsilon) \quad \text{for all the other } n < 0. \tag{153}$$

With these results, we can calculate the value of ν to $O(\epsilon)$, which is given by

$$\begin{aligned}
\nu &= \ell + \frac{1}{2\ell+1}\left[-2 - \frac{s^2}{\ell(\ell+1)} + \frac{[(\ell+1)^2 - s^2]^2}{(2\ell+1)(2\ell+2)(2\ell+3)}\right. \\
&\quad \left. - \frac{(\ell^2 - s^2)^2}{(2\ell-1)2\ell(2\ell+1)}\right]\epsilon^2 + O(\epsilon^3).
\end{aligned} \tag{154}$$

Now one can take the limit of an integer value of s. In particular, the above holds also for $s = 0$. Interestingly, ν is found to be independent of the azimuthal eigenvalue m to $O(\epsilon^2)$.

The post-Minkowski expansion of homogeneous Teukolsky functions can be obtained with arbitrary accuracy by solving Eq. (105) to a desired order and by summing up the terms to a sufficiently large $|n|$. First few terms of the coefficients f^{ν}_n are explicitly given in [43]. A calculation up to a much higher order in $O(\epsilon)$ was performed in [71], in which the black hole absorption of gravitational waves was calculated to $O(v^8)$ beyond the lowest order.

5 Gravitational waves from a particle orbiting a black hole

Based on the ingoing wave functions discussed in the previous two sections, we can derive gravitational wave energy and angular momentum flux emitted to infinity. The formula for the energy and the angular momentum luminosity to infinity are given by Eqs. (35) and (36). Since most of the calculations are very long, we only show the final results. In [46], some details of the calculations are summarized. We define the post-Newtonian expansion parameter by $x \equiv (M\Omega_\varphi)^{1/3}$ where M is the mass of the black hole, Ω_φ is the orbital angular frequency of the particle. Since the parameter x is directly related to the observable frequency, this result can be compared with the results by another method easily.

5.1 Circular orbit around a Schwarzschild black hole

First, we show gravitational waves luminosity emitted by a particle in circular orbit around the Schwarzschild black hole [73, 77]. In this case, Ω_φ is given, in the Schwarzschild coordinate, by $\Omega_\varphi = (M/r_0^3)^{1/2} \equiv \Omega_c$, where r_0 is the orbital

radius. The luminosity to $O(x^{11})$ is given by

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle \;=\; & \left(\frac{dE}{dt} \right)_N \\
& \times \left[1 - \frac{1247}{336}\, x^2 + 4\,\pi\, x^3 - \frac{44711}{9072}\, x^4 - \frac{8191\,\pi}{672}\, x^5 \right. \\
& \quad + \left(\frac{6643739519}{69854400} - \frac{1712\,\gamma}{105} + \frac{16\,\pi^2}{3} - \frac{3424\,\ln 2}{105} - \frac{1712\,\ln x}{105} \right) x^6 \\
& \quad - \frac{16285\,\pi}{504}\, x^7 \\
& \quad + \left(-\frac{323105549467}{3178375200} + \frac{232597\,\gamma}{4410} - \frac{1369\,\pi^2}{126} \right. \\
& \qquad \left. + \frac{39931\,\ln 2}{294} - \frac{47385\,\ln 3}{1568} + \frac{232597\,\ln x}{4410} \right) x^8 \\
& \quad + \left(\frac{265978667519\,\pi}{745113600} - \frac{6848\,\gamma\,\pi}{105} - \frac{13696\,\pi\,\ln 2}{105} - \frac{6848\,\pi\,\ln x}{105} \right) x^9 \\
& \quad + \left(-\frac{2500861660823683}{2831932303200} + \frac{916628467\,\gamma}{7858620} - \frac{424223\,\pi^2}{6804} \right. \\
& \qquad \left. - \frac{83217611\,\ln 2}{1122660} + \frac{47385\,\ln 3}{196} + \frac{916628467\,\ln x}{7858620} \right) x^{10} \\
& \quad + \left(\frac{8399309750401\,\pi}{101708006400} + \frac{177293\,\gamma\,\pi}{1176} \right. \\
& \qquad \left. \left. + \frac{8521283\,\pi\,\ln 2}{17640} - \frac{142155\,\pi\,\ln 3}{784} + \frac{177293\,\pi\,\ln x}{1176} \right) x^{11} \right] \quad (155)
\end{aligned}
$$

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity given by

$$
\left(\frac{dE}{dt} \right)_N = \frac{32\mu^2 M^3}{5 r_0^5} = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^{10}. \qquad (156)
$$

This is the 5.5PN formula beyond the lowest, Newtonian quadrapole formula. We can find that our result agrees with the standard post-Newtonian results up to $O(x^5)$ [14] in the limit $\mu/M \ll 1$.

5.2 Circular orbit on the equatorial plane around a Kerr black hole

Next, we consider a particle in circular orbit on the equatorial plane around a Kerr black hole [74]. In this case, the orbital angular frequency Ω_φ is given by

$$
\Omega_\varphi = \Omega_c \left[1 - q v^3 + q^2 v^6 + O(v^9) \right], \qquad (157)
$$

where Ω_c is the orbital angular frequency of the circular orbit in Schwarzschild case, $v = (M/r_0)^{1/2}$, $q = a/M$, and r_0 is the orbital radius in the Boyer-Lindquist coordinate. The effect of the angular momentum of the black hole is

given by the corrections depending on the parameter q. Here, q is arbitrary as long as $|q| < 1$. The luminosity is given up to $O(x^8)$ (4PN order) by

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle \;=\; & \left(\frac{dE}{dt} \right)_{\mathrm{N}} \left(1 + (q\text{-independent terms}) - \frac{11\,q}{4}\,x^3 + \frac{33\,q^2}{16}\,x^4 \right. \\
& -\; \frac{59\,q}{16}\,x^5 + \left(-\frac{65\,\pi\,q}{6} + \frac{611\,q^2}{504} \right) x^6 \\
& +\; \left(\frac{162035\,q}{3888} + \frac{65\,\pi\,q^2}{8} - \frac{71\,q^3}{24} \right) x^7 \\
& +\; \left. \left(\frac{359\,\pi\,q}{14} + \frac{22667\,q^2}{4536} + \frac{17\,q^4}{16} \right) x^8 \right).
\end{aligned}
\tag{158}
$$

5.3 Slightly eccentric orbit around a Schwarzschild black hole

Next, we consider a particle in slightly eccentric orbit on the equatorial plane around a Schwarzschild black hole ([46],Sec.7). We define r_0 as the minimum of radial potential $R(r)/r^4$. We also define an eccentricity parameter e from the maximum radius of the orbit $r_{\max}$, which is given by $r_{\max} = r_0(1 + e)$. These conditions are explicitly given by

$$
\left. \frac{\partial (R/r^4)}{\partial r} \right|_{r=r_0} = 0, \quad \text{and} \quad R(r_0(1 + e)) = 0.
\tag{159}
$$

We assume $e \ll 1$. In this case, Ω_φ is given to $O(e^2)$ by

$$
\Omega_\varphi = \Omega_c \left[1 - f(v)e^2 \right], \qquad f(v) = \frac{3(1 - 3v^2)(1 - 8v^2)}{2(1 - 2v^2)(1 - 6v^2)},
\tag{160}
$$

where $\Omega_c = (M/r_0^3)^{1/2}$ is the orbital angular frequency in the circular orbit case. We show the energy and angular momentum luminosity which are accurate to $O(e^2)$ and to $O(x^8)$ beyond the Newtonian order. They are given by

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle \;=\; & \left(\frac{dE}{dt} \right)_{\mathrm{N}} \left[1 + (e\text{-independent terms}) \right. \\
& +e^2 \left(\frac{157}{24} - \frac{6781\,x^2}{168} + \frac{2335\,\pi\,x^3}{48} - \frac{14929\,x^4}{189} - \frac{773\,\pi\,x^5}{3} \right. \\
& +\frac{156066596771\,x^6}{69854400} - \frac{106144\,\gamma\,x^6}{315} + \frac{992\,\pi^2\,x^6}{9} \\
& -\frac{80464\,x^6 \ln 2}{315} - \frac{234009\,x^6 \ln 3}{560} - \frac{106144\,x^6 \ln x}{315} \\
& -\frac{32443727\,\pi\,x^7}{48384} - \frac{3045355111074427\,x^8}{671272842240} + \frac{507208\,\gamma\,x^8}{245} \\
& -\frac{31271\,\pi^2\,x^8}{63} - \frac{151336\,x^8 \ln 2}{441} + \frac{12887991\,x^8 \ln 3}{3920}
\end{aligned}
$$

$$+\frac{507208\,x^8\,\ln x}{245}\Big)\Bigg],\tag{161}$$

and

$$\left\langle\frac{dJ_z}{dt}\right\rangle = \left(\frac{dJ}{dt}\right)_{\rm N}\Bigg[1+(e\text{-independent terms})$$

$$+e^2\left(\frac{23}{8}-\frac{3259\,x^2}{168}+\frac{209\,\pi\,x^3}{8}-\frac{1041349\,x^4}{18144}-\frac{785\,\pi\,x^5}{6}\right.$$

$$+\frac{91721955203\,x^6}{69854400}-\frac{41623\,\gamma\,x^6}{210}+\frac{389\,\pi^2\,x^6}{6}-\frac{24503\,x^6\,\ln 2}{210}$$

$$-\frac{78003\,x^6\,\ln 3}{280}-\frac{41623\,x^6\,\ln x}{210}-\frac{91565\,\pi\,x^7}{168}$$

$$-\frac{105114325363\,x^8}{72648576}+\frac{696923\,\gamma\,x^8}{630}-\frac{4387\,\pi^2\,x^8}{18}$$

$$\left.-\frac{7051\,x^8\,\ln 2}{10}+\frac{3986901\,x^8\,\ln 3}{1960}+\frac{696923\,x^8\,\ln x}{630}\right)\Bigg],\tag{162}$$

where $(dJ/dt)_{\rm N}$ is the Newtonian angular momentum flux expressed in terms of x,

$$\left(\frac{dJ_z}{dt}\right)_{\rm N}=\frac{32}{5}\left(\frac{\mu}{M}\right)^2 Mx^7,\tag{163}$$

and the e-independent terms in both $\langle dE/dt\rangle$ and $\langle dJ/dt\rangle$ are the same and are given by the terms in the case of circular orbit, Eq. (155).

5.4   Slightly eccentric orbit around a Kerr black hole

Next, we consider a particle in slightly eccentric orbit on the equatorial plane around a Kerr black hole [70]. We define the orbital radius r_0 and the eccentricity in the same way as in the Schwarzschild case by

$$\left.\frac{\partial(R/r^4)}{\partial r}\right|_{r=r_0}=0,\quad\text{and}\quad R(r_0(1+e))=0.\tag{164}$$

We also assume $e\ll 1$. In this case, Ω_φ is given to $O(e^2)$ by

$$\Omega_\varphi = \Omega_c\Bigg[1-qv^3+e^2\left(-\frac{3}{2}+\frac{9}{2}v^2-\frac{9}{2}qv^3+3\left(6+q^2\right)v^4-60qv^5\right)$$

$$+O(v^6)\Bigg].\tag{165}$$

We show the energy and angular momentum luminosity which are accurate to $O(e^2)$ and to $O(x^5)$ beyond Newtonian order.

$$\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{dE}{dt}\right)_{\rm N}\Bigg\{1-\frac{1247}{336}\,x^2-\frac{11}{4}\,qx^3+4\,x^3\pi-\frac{44711}{9072}\,x^4+\frac{33}{16}\,x^4q^2$$

$$-\frac{59}{16}qx^5 - \frac{8191}{672}x^5\pi + \Big(\frac{157}{24} - \frac{6781}{168}x^2 - \frac{2009}{72}qx^3 + \frac{2335}{48}x^3\pi$$
$$-\frac{14929}{189}x^4 + \frac{281}{16}x^4q^2 - \frac{2399}{56}qx^5 - \frac{773}{3}x^5\pi)e^2\Big\}, \qquad (166)$$

$$\left\langle\frac{dJ_z}{dt}\right\rangle = \left(\frac{dJ_z}{dt}\right)_{\rm N}\Big\{1 - \frac{1247}{336}x^2 - \frac{11}{4}qx^3 + 4\,x^3\pi - \frac{44711}{9072}x^4 + \frac{33}{16}x^4q^2$$
$$-\frac{59}{16}qx^5 - \frac{8191}{672}x^5\pi + \Big(\frac{23}{8} - \frac{3259}{168}x^2 - \frac{371}{24}qx^3 + \frac{209}{8}x^3\pi$$
$$-\frac{1041349}{18144}x^4 + \frac{171}{16}x^4q^2 - \frac{243}{8}qx^5 - \frac{785}{6}x^5\pi)e^2\Big\}. \qquad (167)$$

5.5 Circular orbit with small inclination from the equatorial plane around a Kerr black hole

Next, we consider a particle in circular orbit with small inclination from the equatorial plane around a Kerr black hole [69]. In this case, apart from the energy E and z-component of the angular momentum l_z, the particle motion has another constant of motion; the Carter constant C. The orbital plane of the particle precesses around the symmetric axis of black hole, and the degree of precession is determined by the value of the Carter constant. We introduce a dimensionless parameter y defined by

$$y = \frac{C}{Q^2}, \qquad Q^2 = l_z^2 + a^2(1 - E^2). \qquad (168)$$

Given the Carter constant and orbital radius r_0, the energy and angular momentum is uniquely determined by $R(r) = 0$, and $\partial R(r)/\partial r = 0$. By solving the geodesic equation with the assumption $y \ll 1$, we find that $y^{1/2}$ is equal to the inclination angle from the equatorial plane. The angular frequency Ω_φ is determined to $O(y)$ and $O(v^4)$ as

$$\Omega_\varphi = \Omega_c\left[1 - qv^3 + \frac{3}{2}y\left(qv^3 - q^2v^4\right) + O(v^6)\right]. \qquad (169)$$

We show the energy and the z-component angular flux to $O(v^5)$.

$$\left\langle\frac{dE}{dt}\right\rangle = \left(\frac{dE}{dt}\right)_{\rm N}\left(1 - \frac{1247}{336}v^2 + 4\pi v^3 - \frac{73}{12}qv^3\left(1 - \frac{y}{2}\right) - \frac{44711}{9072}v^4\right.$$
$$\left.+\frac{33}{16}q^2v^4 - \frac{527}{96}q^2v^4y - \frac{8191\pi}{672}v^5 + \frac{3749}{336}qv^5\left(1 - \frac{y}{2}\right)\right). \qquad (170)$$

$$\left\langle\frac{dJ_z}{dt}\right\rangle = \left(\frac{dJ_z}{dt}\right)_{\rm N}\left[\left(1 - \frac{y}{2}\right) - \frac{1247}{336}v^2\left(1 - \frac{y}{2}\right) + 4\pi v^3\left(1 - \frac{y}{2}\right)\right.$$
$$\left.-\frac{61}{12}qv^3\left(1 - \frac{y}{2}\right) - \frac{44711}{9072}v^4\left(1 - \frac{y}{2}\right) + q^2v^4\left(\frac{33}{16} - \frac{229}{32}y\right)\right)$$

40

$$-\frac{8191}{672}\pi v^5\left(1-\frac{y}{2}\right)+qv^5\left(\frac{417}{56}-\frac{4301}{224}y\right)\Bigg]. \tag{171}$$

Using Eq. (169), we can express v in terms of $x=(M\Omega_\varphi)^{1/3}$ as

$$v=x\left[1+\frac{q}{3}x^3+\frac{1}{2}y\left(-qx^3+q^2x^4\right)\right]. \tag{172}$$

We then express Eqs. (170) and (171) in terms of x as

$$
\begin{aligned}
\left\langle\frac{dE}{dt}\right\rangle &= \left(\frac{dE}{dt}\right)_{\mathrm{N}}\left[1-\frac{1247}{336}x^2+\left(4\pi-\frac{11}{4}q-\frac{47}{24}qy\right)x^3\right.\\
&\quad -\frac{44711}{9072}x^4+q^2x^4\left(\frac{33}{16}-\frac{47}{96}y\right)-\frac{8191\,\pi}{672}x^5\\
&\quad \left.+qx^5\left(-\frac{59}{16}+\frac{11215}{672}y\right)\right],
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
\left\langle\frac{dJ_z}{dt}\right\rangle &= \left(\frac{dJ_z}{dt}\right)_{\mathrm{N}}\left[\left(1-\frac{y}{2}\right)-\frac{1247}{336}x^2\left(1-\frac{y}{2}\right)+4\pi x^3\left(1-\frac{y}{2}\right)\right.\\
&\quad -qx^3\left(\frac{7}{2}y+\frac{11}{4}\left(1-\frac{y}{2}\right)\right)-\frac{44711}{9072}x^4\left(1-\frac{y}{2}\right)+q^2x^4\left(\frac{33}{16}-\frac{117}{32}y\right)\\
&\quad \left.-\frac{8191}{672}\pi x^5\left(1-\frac{y}{2}\right)+qx^5\left(-\frac{59}{16}+\frac{687}{224}y\right)\right].
\end{aligned}
\tag{174}
$$

5.6   Absorption of gravitational waves by a black hole

In this section, we evaluate the energy absorption rate by a black hole. The energy flux formula is given by [79]

$$\left(\frac{dE_{\mathrm{hole}}}{dtd\Omega}\right)=\sum_{\ell m}\int d\omega\,\frac{2S_{\ell m}^2}{2\pi}\frac{128\omega k(k^2+4\tilde{\epsilon}^2)(k^2+16\tilde{\epsilon}^2)(2Mr_+)^5}{|C|^2}|\tilde{Z}_{\ell m\omega}^{\mathrm{H}}|^2, \tag{175}$$

where $\tilde{\epsilon}=\kappa/(4r_+)$ and

$$
\begin{aligned}
|C|^2 &= \left((\lambda+2)^2+4a\omega m-4a^2\omega^2\right)\left[\lambda^2+36a\omega m-36a^2\omega^2\right]\\
&\quad +(2\lambda+3)(96a^2\omega^2-48a\omega m)+144\omega^2(M^2-a^2).
\end{aligned}
\tag{176}
$$

In calculating $\tilde{Z}_{\ell m\omega}^{\mathrm{H}}$, we need to evaluate the upgoing solution R^{up}, and asymptotic amplitude of ingoing, and upgoing solution, B^{inc}, B^{trans}, and C^{trans} in Eqs. (14) and (15). Evaluation of the incident amplitude B^{trans} of the ingoing solution is essential in the calculation. Poisson and Sasaki [59] evaluated them, in the case of circular orbit around the Schwarzschild black hole, up to $O(\epsilon)$ beyond the lowest order, and obtained the energy flux at the lowest order, using the method we have described in section 3. Later, Tagoshi, Mano, and Takasugi [71] evaluated the energy absorption rate, in the Kerr case, to $O(v^8)$ beyond the lowest order using the method in section 4. Since the resulting

formula is very long and complicated, we only show here to $O(v^3)$ beyond the lowest order. The energy absorption rate is given by

$$
\begin{aligned}
\left(\frac{dE}{dt}\right)_{\mathrm{H}} &= \frac{32}{5}\left(\frac{\mu}{M}\right)^2 x^{10}\, x^5 \left[-\frac{1}{4}\, q - \frac{3}{4}\, q^3 + \left(-q - \frac{33}{16}\, q^3\right) x^2 \right. \\
&\quad + \left(2\, q B_2 + \frac{1}{2} + \frac{13}{2}\, \kappa\, q^2 + \frac{35}{6}\, q^2 - \frac{1}{4}\, q^4 + \frac{1}{2}\, \kappa \right. \\
&\quad \left. + 3\, q^4 \kappa + 6\, q^3 B_2\right) x^3 \Bigg],
\end{aligned}
\tag{177}
$$

where

$$
B_n = \frac{1}{2i}\left[\psi^{(0)}\left(3 + \frac{niq}{\sqrt{1-q^2}}\right) - \psi^{(0)}\left(3 - \frac{niq}{\sqrt{1-q^2}}\right)\right],
\tag{178}
$$

and $\psi^{(n)}(z)$ is the polygamma function. We see that absorption effect begins at $O(v^5)$ beyond the quadrapole formula in the case $q \neq 0$. If we set $q = 0$ in the above formula, we have

$$
\left(\frac{dE}{dt}\right)_{\mathrm{H}} = \left(\frac{dE}{dt}\right)_N \left(x^8 + O(v^{10})\right),
\tag{179}
$$

which was obtained by Poisson and Sasaki [59].

We note that the leading terms in $(dE/dt)_{\mathrm{H}}$ are negative for $q > 0$, i.e., the black hole loses the energy if the particle is co-rotating. This is because of the superradiance for modes with $k < 0$.

6   Conclusion

In this article, we described analytical approaches to calculate gravitational radiation from a particle of mass μ orbiting a black hole of mass M with $M \gg \mu$, based upon the perturbation formalism developed by Teukolsky. A review on the this formalism was given in section 2. The Teukolsky equation, which governs the gravitational perturbation of a black hole, is too complicated to be solved analytically. Therefore, one has to adopt a certain approximation scheme. The scheme we employed is the post-Minkowski expansion, in which all the quantities are expanded in terms of a parameter $\epsilon = 2M\omega$ where ω is the Fourier frequency of the gravitational waves. For the source term given by a particle in bound orbit, this naturally gives the post-Newtonian expansion.

In section 3, we considered the case of a Schwarzschild background. For a Schwarzschild black hole, one can transform the Teukolsky equation to the Regge-Wheeler equation. The advantage of the Regge-Wheeler equation is that it reduces to the standard Klein-Gordon equation in the flat space limit, and hence it is easier to understand the post-Minkowskian or post-Newtonian effects. Therefore, we adopted this method in the case of a Schwarzschild background. However, the post-Minkowski expansion of the Regge-Wheeler equation is not

quite systematic, and as one goes to higher orders, the equations to be solved become increasingly complicated. Furthermore, for a Kerr background, although one can perform a transformation similar to the Chandrasekhar transformation, it can be done only at the expense of losing the reality of the equation. Thus, the resulting equation is not quite suited to analytical treatments.

In section 4, we described a different method, developed by Mano, Suzuki and Takasugi [45, 43], that directly deals with the Teukolsky equation and considered the case of a Kerr background with this method. Although the method is mathematically rather complicated and it is hard to obtain physical insights into relativistic effects, it has a great advantage that it allows a systematic post-Minkowski expansion of the Teukolsky equation, even on the Kerr background. We gave a thorough review on how this method works and how it gives a systematic post-Minkowski expansion.

Finally, in section 5, we recapitulated the results of calculations of the gravitational waves for various orbits that had been obtained by various authors by the methods described in sections 3 and 4. These results are useful not only by themselves for the actual case of a compact star orbiting a supermassive black hole, but also give us useful insights into higher order post-Newtonian effects even for a system of equal-mass binaries.

7   Acknowledgements

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